Stability and resonances in an oscillator with delayed feedback

Griselda R. Itovich
Esc. de Prod., Tecn. y Medio Amb.
Sede Alto Valle y Valle Medio
Universidad Nacional de Río Negro
Villa Regina - ARGENTINA
Email: gitovich@unrn.edu.ar

Franco S. Gentile
Dpto. de Matemática
Universidad Nacional del Sur (UNS)
IIIE (UNS-CONICET)
Bahía Blanca - ARGENTINA
Email: fsgentile@gmail.com

Jorge L. Moiola
IIIE (UNS-CONICET)
Dpto. de Ing. Eléct. y de Comp. (UNS)
Bahía Blanca - ARGENTINA
Email: jmoiola@uns.edu.ar

***Abstract*—A bidimensional system of delay differential equations, including four parameters, is analyzed. The changes of stability of its equilibrium have been determined clearly through a set of parameters conditions. Otherwise, now by means of the frequency domain approach, the Hopf bifurcation and the stability of the emergent limit cycles have been examined thoroughly.**

***Index Terms*—delay-differential equations, Hopf bifurcation, stability, resonances**

I. Introduction

The estimation of the critical gain and the critical frequency in a feedback system using a simple method is of vital importance, not only concerning the stability issues per se but also improving the automatic tuning of simple regulators [1], [2]. The computation of the oscillation characteristics has been obtained using the approach of relay systems and/or the describing function analysis [3] in order to set later the parameters of the PID controllers under specified rules. In particular, autotuning principles for generalization of PID controllers have been applied also to SISO (single-input to single-output) systems with delays (see, for example, [4] and the references therein).

The determination of the critical gain and the critical frequency, simultaneously, plays a singular role for stability issues when dealing with a nonlinear feedback system since autonomous oscillations are expected to arise under the now classical phenomenon of Hopf bifurcation [5], [6]. The same scenario can be used for special systems having a time-delayed feedback as it was shown in [7]. In this regard, the frontiers of stability are called Hopf bifurcation curves in the space of system parameters. Moreover, taking into account the complexity added by the delays, the characteristic polynomial is now changed as a quasi-polynomial due to the exponential terms [8]. Then and in simple control engineering terms, the complexity of stability analysis is enlarged due to the possibility of multiple root crossing. This is the case of the appearance of resonances since two critical frequencies coincide for the same values of system parameters [9].

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In the present article, the analysis of the regions of stability and the Hopf bifurcation condition, this last using a frequency domain approach [6], [7], is presented. The example has been analyzed previously in [9] using a different method and with less parameters. The results are complementary of those recently shown in [10] where an hybrid technique (an analytical method plus numerical computational one provided by Dde-Biftool [11]) has been successfully applied.

II. Model under analysis

The aim of this work is the analysis of the following delay-differential equation

$$\ddot{x} + \gamma x = f(\dot{x}(t-\tau)), \tag{1}$$

where $f(x) = \delta x + \eta x^2$ is the feedback action depending on the velocity, with $\gamma, \tau > 0$ and $\delta, \eta \neq 0$. Thus, four real parameters are involved and there is a single equilibrium point: $x = 0$. This equation comes from a similar one in [9] but it is explored with different techniques, just to discover and clarify some dynamic aspects. As (1) is a delay-differential equation (see [12]), it can also be written as a delay-differential system in $X = (x_1, x_2)$ as

$$\begin{aligned} \dot{x}_1 &= -\gamma x_2 + f(x_1(t-\tau)), \\ \dot{x}_2 &= x_1, \end{aligned} \tag{2}$$

with equilibrium $X_0 = (x_1, x_2) = (0, 0)$. To explore its stability, one needs to consider the linearization of system (2) about X_0, given by

$$\dot{u} = Au + A_\tau u(t-\tau), \tag{3}$$

where

$$A = \begin{pmatrix} 0 & -\gamma \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad A_\tau = \begin{pmatrix} \delta & 0 \\ 0 & 0 \end{pmatrix}.$$

To guarantee the asymptotic stability of the equilibrium, the characteristic equation $\det\left(\lambda I - A - A_\tau e^{-\lambda\tau}\right) = 0$, i.e.

$$\lambda^2 - \delta e^{-\lambda\tau}\lambda + \gamma = 0 \Leftrightarrow e^{\lambda\tau}\left(\lambda^2 + \gamma\right) - \delta\lambda = 0, \tag{4}$$

must have all its roots lying on the left half plane. Through a change of variables $v = \lambda\tau$ in (4), results

$$H(v) = h(v, e^v) = e^v(v^2 + \gamma\tau^2) - \delta\tau v = 0. \tag{5}$$

It must be observed that H is an exponential polynomial. In this context, one needs the definition of what it is called its principal term.

Definition 1: Let be $h(z, e^z) = \sum_{m,n} a_{mn} e^{zm} z^n$ an exponential polynomial. The term $a_{rs} e^{zr} z^s$ is called the principal term of h if $a_{rs} \neq 0$ and, if for each other term $a_{mn} e^{zm} z^n$ with $a_{mn} \neq 0$, it is satisfied $r > m, s > n$, or $r = m, s > n$, or $r > m, s = n$.

It can be proved that if an exponential polynomial has no principal term, then it has an unbounded number of ceros with arbitrarily large real part [8], e.g. $P(z) = e^z - z$. Then, it must be observed that H (see (5)) has a principal term which results $e^v v^2$.

The analysis of the roots of H is given by means of the next theorem [8]:

Theorem 1: Let be $H(v) = h(v, e^v)$ where h is a polynomial with a nonzero principal term. It is written $H(iy) = F(y) + iG(y)$, $y \in \mathbb{R}$. If all the roots of the function F are real and for each of these zeros the condition $F'(y)G(y) < 0$ is satisfied, then all the zeros of H are located on the left of the imaginary axis.

Remark 1: The condition $F'(y)G(y) < 0$ means that $G(y) \neq 0$ for each root of F, that in addition must be all simple (to carry out $F'(y) \neq 0$).

By (5), the expression $H(iy) = F(y) + iG(y)$ leads to

$$\begin{aligned} F(y) &= \ (-y^2 + \gamma\tau^2)\cos y, \\ G(y) &= \ (-y^2 + \gamma\tau^2)\sin y - \delta\tau y. \end{aligned} \tag{6}$$

Notice that the condition of being $F(y)$ and $G(y)$ simultaneously zero is equivalent to state that H has a pair of imaginary roots.

In relation with the *Remark 1*, some necessary conditions on the involved parameters are deduced, just to satisfy the hypotheses of Theorem 1.

The zeros of F are

$$\text{a)} \ \ y_k = (2k-1)\,\frac{\pi}{2},\ k \in \mathbb{Z}, \qquad \text{b)} \ \ \breve{y}_{1,2} = \pm\gamma^{0.5}\tau. \tag{7}$$

Then:

a) As one needs that F and G do not share any of their roots, it is required $G(y_k) \neq 0$. This implies that the chosen values of the parameters γ, τ and δ in (5) should not satisfy the equation

$$(-1)^k (y_k^2 - \gamma\tau^2) - \delta\tau y_k = 0, \quad k \in \mathbb{Z}. \tag{8}$$

b) As $\gamma, \tau > 0$ and $\delta \neq 0$ from the beginning, it is always truth that

$$G(\breve{y}_{1,2}) = \mp\delta\gamma^{0.5}\tau^2 \neq 0.$$

Remark 2: When (8) is satisfied, the roots of (4) are on the imaginary axis. This condition determines the Hopf surfaces in the three dimensional parameter space $\gamma - \tau - \delta$ (for system (2)). Or, fixing the variable γ, the Hopf curves δ_k in the $\tau - \delta$ plane are determined by

$$\delta = \delta_k(\tau) = (-1)^k \left(\frac{y_k}{\tau} - \frac{\gamma\tau}{y_k} \right). \tag{9}$$

Thus, it is possible to set certain conditions to guarantee that the roots of F are not of G's.

Now, to ensure that the roots of F are simple, it is also required that

$$\gamma\tau^2 \neq (2k-1)^2 \left(\frac{\pi}{2}\right)^2,\ k \in \mathbb{Z}, \tag{10}$$

to fulfill $F'(y) \neq 0$, for any root y of F.

Thereby, taking into account the case $\gamma = 1$, due to (10), one knows that τ satisfies one of these two conditions: 1) $0 < \tau < y_1$ or 2) $y_k < \tau < y_{k+1}$, $k \in \mathbb{N}$. Henceforth, just to simplify the next exposition of results, γ is fixed as 1. Thus, the following theorem can be set:

Theorem 2: It is considered (5) and (6) with $\gamma = 1, \tau > 0, \delta \neq 0$. Let be $y_k, k \in \mathbb{N}$ and $\breve{y}_{1,2}$ given by (7). Assume that the parameters values do not satisfy (8). Then all the roots of H lie on the left half plane if these parameters conditions are fulfilled:

I) For $0 < \tau < y_1$, it must be

$$\delta > -\frac{y_1}{\tau} + \frac{\tau}{y_1}, \quad \delta < \frac{y_2}{\tau} - \frac{\tau}{y_2}, \quad \delta < 0.$$

II) For $y_k < \tau < y_{k+1}, k \in \mathbb{N}$, it is required:

a) If k is odd, then

$$\delta < -\frac{y_k}{\tau} + \frac{\tau}{y_k}, \quad \delta < \frac{y_{k+1}}{\tau} - \frac{\tau}{y_{k+1}}, \quad \delta > 0.$$

b) If k is even, then

$$\delta > \frac{y_k}{\tau} - \frac{\tau}{y_k}, \quad \delta > -\frac{y_{k+1}}{\tau} + \frac{\tau}{y_{k+1}}, \quad \delta < 0.$$

Proof (Sketch): In order to apply 1, it is necessary to check that $F'(y)G(y) < 0$ for each y, an arbitrary root of F (see (7)).

I) The first two inequalities result analyzing the necessary statements to fulfill $F'(y_1)G(y_1) < 0$ and $F'(y_2)G(y_2) < 0$. To satisfy $F'(y_k)G(y_k) < 0$, $k > 2$, it can be shown that the obtained conditions are sufficient, by considering separately the cases k odd or k even. The sign of δ follows from the requirement of $F'(\breve{y}_i)G(\breve{y}_i) < 0, i = 1, 2$.

II) The first two inequalities in a) and b) are deduced imposing $F'(y_k)G(y_k) < 0$ and $F'(y_{k+1})G(y_{k+1}) < 0$, when k is odd or even. These conditions result sufficient to guarantee $F'(y_i)G(y_i) < 0$, where $i < k$ or $i > k$. To show this last it is necessary to consider the four different cases which result for i odd or even. Finally, the sign of δ is established from $F'(\breve{y}_i)G(\breve{y}_i) < 0, i = 1, 2$, according to k being odd or even.

The details to prove the Case II)a) appear in the Appendix at the end of this work. The other cases are similar.

It still remains to test that $F'(y_k)G(y_k) < 0$ for each y_k, a root of F where $y_k < 0$. Given that $F'(-y_k)G(-y_k) = F'(y_k)G(y_k)$, now the proof is complete.

Corollary 1: The equilibrium of equation (1) or system (2) results asymptotically stable under the parameter conditions established in Theorem 2.

Remark 3: Theorem 2 can be stated for an arbitrary $\gamma > 0$ in a similar way.

Remark 4: An equivalent result can be established by the assignment of a fixed value for τ in (5) and (6). In this case, the resulting restrictions between γ and δ are always linear. Considering $\tau = \pi$ in (9), the Hopf curves become $\bar{\alpha}_k = (-1)^k(-\frac{2k-1}{2} - \frac{2\gamma}{2k-1})$ where $k \in \mathbb{N}$. Related with this assumption, the complete stability analysis of a general delay differential equation which includes (3) can be found in [13].

Due to the previous results, Fig. 1 shows a few colored stability areas, some Hopf curves δ_k (see (9)) and multiple resonant points in the $\tau - \delta$ plane, for the particular case with $\gamma = 1$. The Hopf curves δ_k intersect the τ axis ($\delta = 0$) at the points $\tau = y_k$. The crossings between the curves δ_k and δ_j, $k < j$ (k and j having different parity) are $2k-1 : 2j-1$ resonant points and their coordinates are

$$(\tau, \delta) = \frac{1}{\sqrt{y_k y_j}} \left(y_k y_j, (-1)^k (y_k - y_j) \right),$$

where $y_k < y_j$. The curves δ_k and δ_j do not intersect each other if k and j have the same parity. At the resonance points, (5) has two pair of purely imaginary solutions, namely $\pm i\, y_k/\tau$, $\pm i\, y_j/\tau$ satisfying $y_k/y_j = (2k-1)/(2j-1)$. This situation is related with the interaction between two limit cycles, whose frequencies are $(2k-1)\pi/(2\tau)$ and $(2j-1)\pi/(2\tau)$ approximately. Thereby, the intersection between the Hopf curves δ_1 and δ_2 results a $1:3$ resonant point.

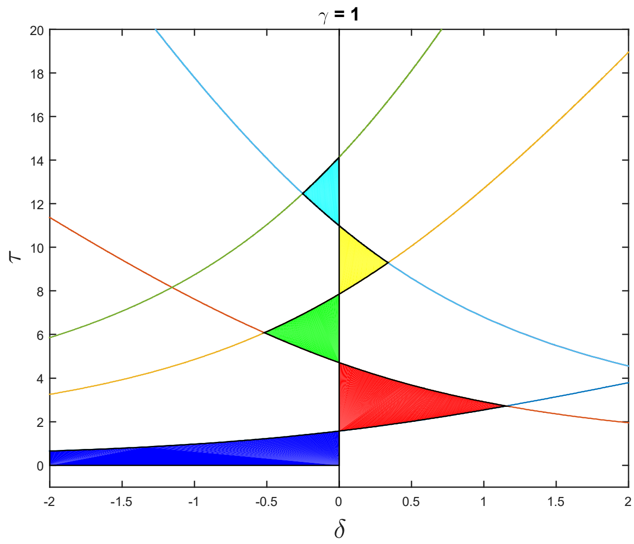

Fig. 1. Stability regions for the equilibrium point of (1) or (2) with $\gamma = 1$. According to 2, regions in red and yellow correspond to $k = 1, 3$ (II)a) odd) respectively, areas in green and cyan refer to $k = 2, 4$ (II)b) even) and finally the blue domain represents the case set in I).

III. Stability About Hopf Curves: A Frequency-Domain Approach

Quantitative and qualitative features about a limit cycle emerging from a Hopf bifurcation point can be analyzed through the frequency domain methodology (see [5], [6] and [7]). Particularly, the information about its stability can be obtained somehow by the computation of the curvature coefficient of the Hopf branch and the analysis of its sign.

It is proposed a feedback representation of the system (2) like

$$\begin{cases} \dot{X} = & A(X) + Bg(c(t-\tau)), \\ g(c) = & f(-c) = -\delta c + \eta c^2, \quad c = -CX, \end{cases}$$

where

$$A = \begin{pmatrix} 0 & -\gamma \\ 1 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad C = B^T.$$

Thus the transfer function for the linear part becomes

$$G^*(s) = C(sI - A)^{-1} B\, e^{-s\tau} = \frac{s e^{-s\tau}}{s^2 + \gamma}. \tag{11}$$

The equilibrium in the frequency domain, $c = 0$, comes from solving the equation $-c = G^*(0)g(c)$. The expression of the characteristic eigenvalue, which equals the (linearized) loop gain, is

$$\lambda = G^* J = -\frac{\delta s e^{-s\tau}}{s^2 + \gamma},$$

where $J = g'(0) = -\delta$. The well known critical condition at the Hopf bifurcation point, i.e. $\lambda(i\omega) = -1$, gives this system of equations

$$\omega^2 - \gamma + \delta\omega \sin\omega\tau = 0, \quad \delta\omega\cos\omega\tau = 0.$$

Then as $\omega \neq 0$ and $\delta \neq 0$ it should be

$$\omega\tau = \frac{(2k-1)}{2}\pi = y_k, \quad k \in \mathbb{N},$$

(due to $\omega\tau > 0$) as well as

$$\omega^2 - \gamma - \delta\omega(-1)^k = 0, \tag{12}$$

which resumes the Hopf bifurcation point condition and results equivalent to (9) for $k \in \mathbb{N}$ after replacing $\omega = \frac{y_k}{\tau}$. Moreover, in this context the formulae to compute the curvature coefficient is

$$\sigma(\omega) = -\text{Re}\left(\frac{G^*(i\omega) p(i\omega)}{G^{*\prime}(i\omega) J} \right). \tag{13}$$

It is necessary to obtain p. In that sense and according to the methodology (see [5])

$$p(i\omega) = D_2 \left(\frac{1}{2} \bar{v} V_{22} + v V_{02} \right) + D_3 v^2 \bar{v},$$

but $v = 1, D_2 = g''(0) = 2\eta$ and $D_3 = g'''(0) = 0$, so it remains to get only V_{02} and V_{22}. In engineering terms, V_{02} is related with the amplitude of the bias for the correction of the equilibrium point due to the nonlinearities; V_{22} is connected to the amplitude of the second harmonic of the oscillatory solution and $p(i\omega)$ is associated to the amplitude of the first harmonic of the periodic solution. Then, as the closed-loop transfer function is

$$H = (1 + G^* J)^{-1} G^*,$$

one has

$$H(0) = 0, \quad H(i2\omega) = -\frac{i2\omega}{(\gamma - 4\omega^2) + \delta i 2\omega},$$

and finally

$$\begin{aligned} V_{02} = & \; -\tfrac{1}{4} H(0) D_2 = 0, \\ V_{22} = & \; -\tfrac{1}{4} H(i2\omega) D_2 = \frac{i\eta\omega}{(\gamma - 4\omega^2) + \delta i 2\omega}. \end{aligned}$$

Provided that $p(i\omega) = \eta V_{22}$, now (see (13)) it is defined

$$aux = G^*(i\omega) p(i\omega) = -\frac{\eta^2 \omega^2 e^{-i\omega\tau}}{(\gamma - \omega^2)(\gamma - 4\omega^2 + \delta i 2\omega)}. \quad (14)$$

Notation: In what follows, the expression $aux(k)$ will mean $aux(i\omega_k)$ where $\omega_k \tau = \frac{(2k-1)}{2}\pi$, e.g. $aux(1)$ denotes $aux(i\omega_1)$ where $\omega_1 \tau = \frac{\pi}{2}$. The same notation is extended for $G^{*\prime}(k)$ and σ. Besides, the subscript o or e means k odd or even respectively.

In agreement with the previous section, only system (2) with $\gamma = 1$ is taken into account. To get the expression of σ, the condition (12) is required and two possibilities must be considered: $\omega_k \tau = \frac{(2k-1)}{2}\pi$, where k is odd or even. Then, the whole analysis gives place to three different situations that will be developed below and thus the stability analysis will be finished.

Now, the sign of the curvature coefficient will be determined for the two cases mentioned in Theorem 2.

Case I) First, from (14), (12) and $\gamma = 1$, for an arbitrary odd value of k results

$$aux_o(k) = \frac{\eta^2 i}{\delta(-3\omega_k + \delta(1+2i))},$$

then if $k = 1$ one has

$$aux_o(1) = \frac{2\tau\eta^2 i}{\delta(-3\pi + 2\tau\delta(1+2i))}.$$

Due to (11), it results

$$G^{*\prime}(s) = \frac{e^{-s\tau}((1 - s\tau)(s^2+1) - 2s^2)}{(s^2+1)^2}$$

and

$$G_o^{*\prime}(1) = -i\frac{(-1 - i\frac{1}{2}\pi)\delta\pi 2\tau + 8\tau^2}{\delta^2 \pi^2}.$$

In agreement with (13), for $\omega\tau = \frac{\pi}{2}$, i.e. $k = 1$, and over the Hopf curve δ_1 follows

$$\sigma(1) = -\mathrm{Re}\left(\frac{aux_o(1)}{-\delta G_o^{*\prime}(1)}\right),$$

and after some tedious calculations leads to $\mathrm{sgn}(\sigma(1)) = -\mathrm{sgn}(P_1)$, where

$$P_1 = 2\pi\tau(\pi - 1)\delta^2 + \left(8\tau^2 + 3\pi^2\right)\delta - 12\pi\tau. \quad (15)$$

Moreover, to find where $\sigma = \sigma(1) = 0$, the system of equations formed by $P_1 = 0$ and $\delta = \delta_1(\tau) = -\frac{\pi}{2}\frac{1}{\tau} + \frac{2}{\pi}\tau$ must be solved. So the unique point of δ_1 where $\sigma = 0$ results $Q = (\tau, \delta) = (2.3979, 0.8715)$. If $\tau < 2.3979$ (or $\delta < 0.8715$) due to $\sigma > 0$, the emergent cycles are unstable. Close to Q, branches of periodic solutions exhibit the cycle folds, where the cycles pass from unstable to stable. On the contrary, for $2.3979 < \tau < \sqrt{y_1 y_2} = \frac{\sqrt{3}\pi}{2}$, as $\sigma < 0$ the cycles result stable (classic Hopf branch). Otherwise, for $\tau > \frac{\sqrt{3}\pi}{2}$, the emergent orbit are unstable. Fig. 2 shows two branches of periodic solutions that are born close to Q, according with both situations described above. These continuations have been obtained using the software package Dde-Biftool [11].

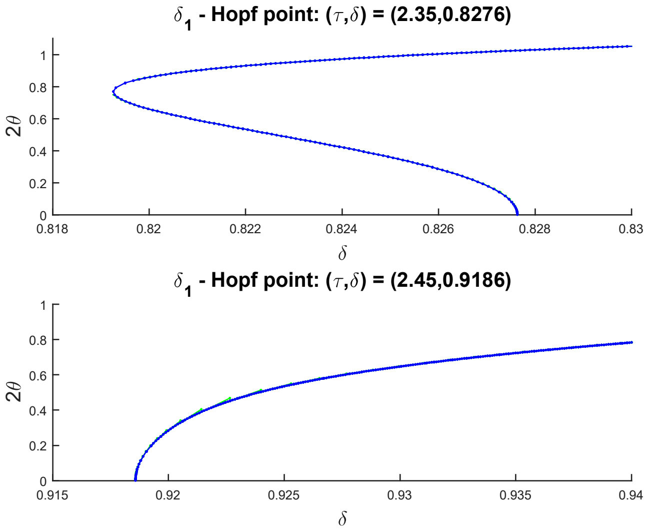

Fig. 2. Branches of periodic solutions of system (2), with $\gamma = \eta = 1$, born at δ_1- Hopf points, close to $Q = (\tau, \delta) = (2.3979, 0.8715)$. (Above: Unstable branch with fold of limit cycles, Below: Stable branch). θ represents an aproximation of the amplitude of the cycle.

Besides, if one replaces $\delta\tau = -\frac{\pi}{2} + \frac{2}{\pi}\tau^2$ into τP_1 (see (15)) and then multiplies by 2, it is obtained another expression $\tilde{P}_1$ in the variable τ that can be written as

$$\tilde{P}_1(\tau) = 2^4\left(1 + \pi^{-1}\right)\tau^4 - 2^2\pi\left(2\pi + 3\right)\tau^2 + \pi^3(\pi - 2^2).$$

Then, applying Descartes's Rule it can be shown that $\tilde{P}_1$ has exactly one positive root τ, as has been commented before.

Case II)a) Now, it is analyzed the case with k odd but $k \geq 3$, remaking the steps done before for $\omega_k \tau = \frac{(2k-1)}{2}\pi = \frac{a}{2}\pi$, where $a = 2k - 1 \geq 5$ if $k \geq 3$. Furthermore, due to (12), one has $\omega^2 - 1 + \delta\omega = 0$ and then

$$aux_o(k) = \frac{2\tau\eta^2 i}{\delta(-3a\pi + 2\tau\delta(1+2i))},$$

as well as

$$G_o^{*\prime}(k) = -i\frac{(-1 - i\frac{(2k-1)}{2}\pi)\delta(2k-1)\pi 2\tau + 8\tau^2}{\delta^2(2k-1)^2\pi^2}.$$

Then for an arbitrary k odd with $k \geq 3$ from

$$\sigma(k) = -\mathrm{Re}\left(\frac{aux_o(k)}{-\delta G_o^{*\prime}(k)}\right),$$

follows $\mathrm{sgn}(\sigma(k)) = -\mathrm{sgn}(P_k)$, where

$$P_k = [2\pi\tau a(\pi a - 1)]\,\delta^2 + \left(8\tau^2 + 3\pi^2 a^2\right)\delta - 12\pi\tau a. \quad (16)$$

Once again, to find the point along the Hopf curve where $\sigma = 0$, it must be solved $P_k = 0$ and $\delta = \delta_k(\tau) = -\frac{a\pi}{2}\frac{1}{\tau} + \frac{2}{a\pi}\tau$. To locate the intersections one can proceed as in the case with $k = 1$. Nevertheless, if k is odd with $k \geq 3$ there exist two solutions. This result also comes out substituting $\tau\delta = -\frac{a\pi}{2} + \frac{2}{a\pi}\tau^2$ into τP_k (see (16)) and then multiplying by 2. Thus, it is attained $P_{(o)k}$ which has the general form:

$$P_{(o)k} = R_o\tau^4 + S_o\tau^2 + T_o,$$

where

$$\begin{aligned} R_o &= 2^4\left(1 + (\pi a)^{-1}\right), \; S_o = -2^2\pi a(2\pi\alpha + 3), \\ T_o &= \pi^3 a^3(\pi a - 2^2). \end{aligned}$$

Given that $P_{(o)k} = 0$ is a biquadratic equation, it can be solved analytically and results

$$\tau^2 = \frac{\pi^2 a^2(2\pi a + 3 \pm \sqrt{25 + 24\pi a})}{2^3\,(\pi a + 1)}.$$

Then τ^2 takes two different positive values only if $\pi a > 4$ but this is satisfied due to $a \geq 5$. So, when $k = 3$, the roots are $\tau_1 = 5.1497$, $\tau_2 = 10.0271$ and the corresponding δ values are -0.869453 and 0.4934145. Thus, considering the curve δ_3 and the regions of stability of the equilibrium point (see Fig. 1), the emergent orbits result stable only for $\sqrt{y_2 y_3} = \frac{\sqrt{15}\pi}{2} < \tau < \frac{\sqrt{35}\pi}{2} = \sqrt{y_3 y_4}$. This outcome can be generalized for any Hopf curve δ_k, with k odd, $k \geq 3$, for the interval $I = \left(\sqrt{y_{k-1}y_k}, \sqrt{y_k y_{k+1}}\right)$.

In summary, it has been shown that if $k = 1, \delta_1$ has a unique point where the curvature coefficient vanishes whereas if k is odd and $k \geq 3$ then δ_k has exactly two points where $\sigma = 0$. This result is original in system (2) and appears after considering the variation of several parameters.

Case II)b) Finally if k is even, taking into account that $\omega_k\tau = \frac{(2k-1)}{2}\pi = \frac{a}{2}\pi$, where $a \geq 3$, due to (14) and (12) results

$$aux_e(k) = \frac{2\tau\eta^2 i}{\delta(-3a\pi + 2\tau\delta(-1 + 2i))},$$

$$G_e^{*\prime}(k) = i\frac{(1 + i\frac{a}{2}\pi)\delta a\pi 2\tau + 8\tau^2}{\delta^2 a^2\pi^2}.$$

In general, for even k, as

$$\sigma(k) = -\mathrm{Re}\left(\frac{aux_e(k)}{-\delta G_e^{*\prime}(k)}\right),$$

then $\mathrm{sgn}(\sigma(k)) = -\mathrm{sgn}\left(P_k\right)$, where

$$P_k = [2\pi\tau a(\pi a + 1)]\,\delta^2 + \left(8\tau^2 + 3\pi^2 a^2\right)\delta + 12\pi\tau a = 0. \quad (17)$$

Making the substitution $\tau\delta = \frac{a\pi}{2} - \frac{2}{a\pi}\tau^2$ into τP_k (17) follows

$$P_{(e)k}(\tau) = R_e\tau^4 + S_e\tau^2 + T_e. \quad (18)$$

where

$$\begin{aligned} R_e &= 2^4\left(1 - (\pi a)^{-1}\right), \; S_e = -2^2\pi a\,(2\pi a - 3), \\ T_e &= \pi^3 a^3\left(\pi a + 2^2\right). \end{aligned}$$

Thus, the equation to solve is

$$R_e\tau^4 + S_e\tau^2 + T_e = 0,$$

but now

$$\tau^2 = \frac{\pi^2 a^2(2\pi a - 3 \pm \sqrt{25 - 24\pi a})}{2^3\,(\pi a - 1)},$$

and the values of τ^2 result complex if $\pi a > \frac{25}{24}$, and this condition is satisfied due to $\pi a > 9$ if $a \geq 3$.

Then, the roots of (18) are not real and definitely (17) has no positive solutions.

Thereby and in brief, if k is even then the Hopf curve δ_k has no points where $\sigma = 0$. Moreover, as σ is always negative along δ_k, when k is even, taking into account the regions of stability of the equilibrium point (see Fig. 1), the emergent orbits are stable specifically only if $\tau \in I$.

In agreement with these computations, Fig. 3 shows some representative results about stability over a few Hopf curves.

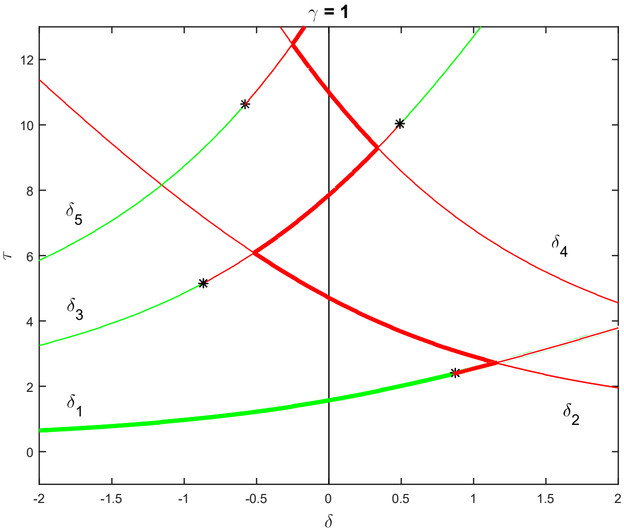

Fig. 3. Some Hopf curves (9) of system (1) with $\gamma = \eta = 1$, are drawn in colors according to the obtained results for $\mathrm{sgn}(\sigma(k))$, where red and green denote negative and positive values respectively. Colored thick lines show stability of emergent limit cycles (red - stable, green - unstable), due to a change of stability of the equilibrium point (see Fig. 1) . The asterisks correspond to the located points over the curves δ_1, δ_3 and δ_5 (one in δ_5 is not shown) where σ vanishes.

Remark 5: The outcomes achieved in this Section for system (2) with $\gamma = 1$, can also be generalized for an arbitrary positive value of γ.

IV. Conclusions

A multiparameter second order delay differential equation, which has a feedback action depending on the velocity was analyzed by means of a combination of techniques. The asymptotic stability of the equilibrium point was established precisely through a system of restrictions on certain parameters. Moreover, the frequency domain methodology has been applied to explore the phenomena of Hopf bifurcation. Thus, the stability of the emergent solutions was completely determined as well as certain singularities related with fold cycle bifurcations were located. It remains to explore the unfolding of certain resonances that were found, like $1:3$, and evaluate the dynamic changes of the model considering other feedback actions of interest.

REFERENCES

[1] K. J. Aström and T. Hägglund, "Automatic tuning of simple regulators with specifications on phase and amplitude margins," *Automatica*, 20(5), 645-651, 1984.

[2] T. S. Schei, "Automatic tuning of PID controllers based on transfer function estimation," *Automatica*, 30(12), 1983-1989, 1994.

[3] D. P. Atherton, *Nonlinear Control Engineering - Describing Function Analysis and Design*, Van Nostrand Reinhold, London, 1975.

[4] R. Prokop, J. Korbel and R. Matušů, "Autotuning principles for time-delay systems," *WSEAS Transactions on Systems*, 11(10), 561-570, 2012.

[5] A. I. Mees and L. Chua, "The Hopf bifurcation theorem and its applications to nonlinear oscillations in circuits and systems," *IEEE Transactions on Circuits and Systems*, 26(4), 235-254, 1979.

[6] A. I. Mees, *Dynamics of Feedback Systems*, John Wiley & Sons, Chichester, UK, 1981.

[7] J. L. Moiola and G. Chen, *Hopf Bifurcation Analysis: A Frequency-Domain Approach*, Vol. 21, World Scientific, Singapore, 1996.

[8] R. Bellman and K. Cooke, *Differential-Difference Equations*, Academic Press, New York, 1963.

[9] S. A. Campbell and J. Bélair, "Resonant codimension two bifurcation in the harmonic oscillator with delayed forcing," *Canadian Applied Mathematics Quarterly*, 7(3), 218-238, 1999.

[10] F. S. Gentile, G. R. Itovich and J. L. Moiola, "Resonant 1:2 double Hopf bifurcation in an oscillator with delayed feedback," *Nonlinear Dynamics*, 91(3), 1779-1789, 2018.

[11] K. Engelborghs, T. Luzyanina, and D. Roose, "Numerical bifurcation analysis of delay differential equations using DDE-BIFTOOL," *ACM Transactions on Mathematical Software*, 28(1), 1-21, 2002.

[12] J. Hale and S. Verduyn Lunel, *Introduction to Functional Differential Equations*, AMS 99, Springer, New York, 1993.

[13] G. Stépán, *Retarded Dynamical Systems: Stability and Characteristic Functions*, Longman, UK, 1989.

APPENDIX

Theorem 2: Proof: Case II)a)

It is necessary to set conditions to satisfy $F'(y)G(y)<0$ for any root y of F given by (7). Let $0<y_k<\tau<y_{k+1}$, $y_k=(2k-1)\frac{\pi}{2}$, $y_{k+1}=(2k+1)\frac{\pi}{2}$ and $\gamma=1$, where k is odd.

1) Analysis with the roots y_k. Through (6), one can find F', then evaluating for y_k one has

$$\begin{aligned} F'(y_k) &= (-1)^k(-y_k^2+\tau^2), \\ G(y_k) &= (-1)^{k+1}(-y_k^2+\tau^2)-\delta\tau y_k. \end{aligned} \tag{19}$$

As k is odd and $F'(y_k)=y_k^2-\tau^2<0$, to fulfill $F'(y_k)G(y_k)<0$ it must be

$$G(y_k)=-y_k^2+\tau^2-\delta\tau y_k>0 \Leftrightarrow \delta<-\frac{y_k}{\tau}+\frac{\tau}{y_k}.$$

Now, as $F'(y_{k+1})=-y_{k+1}^2+\tau^2<0$ and one needs $F'(y_{k+1})G(y_{k+1})<0$ follows

$$G(y_{k+1})=y_{k+1}^2-\tau^2-\delta\tau y_{k+1}>0 \Leftrightarrow \delta<\frac{y_{k+1}}{\tau}-\frac{\tau}{y_{k+1}}.$$

Next, it is considered the same situation with the other roots y_i, y_j such that $0<y_i<y_k<\tau<y_{k+1}<y_j$.

i) Analysis for y_i: It is necessary to take into account the cases where i is odd or even.

- i odd: By means of (19), as $F'(y_i)=y_i^2-\tau^2<0$, it should be $G(y_i)=-y_i^2+\tau^2-\delta\tau y_i>0 \Leftrightarrow \delta<-\frac{y_i}{\tau}+\frac{\tau}{y_i}$. The last inequality is valid due to
$$\delta<-\frac{y_k}{\tau}+\frac{\tau}{y_k}<-\frac{y_i}{\tau}+\frac{\tau}{y_i}.$$
- i even: Through (19) results $F'(y_i)=-y_i^2+\tau^2>0$ so it must be
$$G(y_i)=y_i^2-\tau^2-\delta\tau y_i<0 \Leftrightarrow \delta>\frac{y_i}{\tau}-\frac{\tau}{y_i}. \tag{20}$$
In this case, the strategy for the proof is different. Suppose that the parameters in system (2) belong to the $\tau-\delta$ area Ω defined by $\delta<-\frac{y_k}{\tau}+\frac{\tau}{y_k}$, $\delta<\frac{y_{k+1}}{\tau}-\frac{\tau}{y_{k+1}}$, $\delta>0$, as has been settled in Theorem 2 for the Case II)a). This area is like the region in yellow in Fig. 1, considering $k=3$. Then it remains to prove (20). Consider the function $S(\tau,\delta)=\delta-\left(\frac{y_i}{\tau}-\frac{\tau}{y_i}\right)$. Observe that S has no relative extreme values in the interior of Ω, so its extreme values are on its boundary $\partial\Omega$. Hence, one evaluates S on $\partial\Omega$, where $y_k<\tau<y_{k+1}$ and besides $y_i<y_k$. Thus
$$S\left(\tau,-\frac{y_k}{\tau}+\frac{\tau}{y_k}\right)=(y_k+y_i)\left(-\frac{1}{\tau}+\frac{\tau}{y_k y_i}\right)>0,$$
also
$$S\left(\tau,\frac{y_{k+1}}{\tau}-\frac{\tau}{y_{k+1}}\right)=(y_{k+1}-y_i)\left(\frac{1}{\tau}+\frac{\tau}{y_{k+1}y_i}\right)>0,$$
and at last
$$S(\tau,0)=-\left(\frac{y_i}{\tau}-\frac{\tau}{y_i}\right)=\frac{1}{\tau y_i}\left(-y_i^2+\tau^2\right)>0,$$
as $y_i<y_k<\tau<y_{k+1}$. Then, it can be asserted that S is always positive in the interior of Ω, i.e. $\delta>\frac{y_i}{\tau}-\frac{\tau}{y_i}$ for any point $(\tau,\delta)\in\Omega$. QED.

ii) Analysis for y_j: where $y_k<\tau<y_{k+1}<y_j$. Again, two situations must be considered for j: even or otherwise odd.

- j even: Taking into account (19) results $F'(y_j)=-y_j^2+\tau^2<0$, and this implies $G(y_j)=y_j^2-\tau^2-\delta\tau y_j>0 \Leftrightarrow \delta<\frac{y_j}{\tau}-\frac{\tau}{y_j}$. This last condition is satisfied due to
$$\delta<\frac{y_{k+1}}{\tau}-\frac{\tau}{y_{k+1}}<\frac{y_j}{\tau}-\frac{\tau}{y_j}.$$
- j odd: The proof is similar to the case for y_i with i even, detailed above.

2) Analysis for the roots $\breve{y}_{1,2}=\pm\tau$ (see (7), with $\gamma=1$).

This time, $F'(\breve{y}_{1,2})=-2(\pm\tau)\cos(\pm\tau)$ and $G(\breve{y}_{1,2})=-\delta\tau(\pm\tau)$, then the stability condition $F'(y)G(y)<0$ for the zeros of F, becomes

$$F'(\breve{y}_{1,2})G(\breve{y}_{1,2})=2\delta\tau^3\cos\tau<0.$$

For $y_k<\tau<y_{k+1}$, $y_k=(2k-1)\frac{\pi}{2}$, with k odd, as $\cos\tau<0$ then it must be $\delta>0$.■