A NOTE ON A MATRIX VERSION OF GRÜSS INEQUALITY

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Abstract. In [P. Renaud, *A matrix formulation of Grüss inequality*, Linear Algebra Appl. **335** (2001), 95–100] it was proved an operator inequality involving the usual trace functional. In this article, we give a refinement of such result and we answer positively the Renaud's open problem.

1. Introduction

In 1935, Grüss [5] obtained the following inequality if f, g are integrable real functions on $[a, b]$ and there exist real constant $\alpha, \beta, \gamma, \delta$ such that $\alpha \leq f(x) \leq \beta, \gamma \leq g(x) \leq \delta$ for all $x \in [a, b]$ then

$$\left| \frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2} \int_a^b f(x)dx \int_a^b g(x)dx \right| \leq \frac{1}{4}(\beta - \alpha)(\delta - \gamma), \quad (1.1)$$

and the inequality is sharp, in the sense that the constant $\frac{1}{4}$ cannot be replaced by a smaller one. This inequality has been investigated, applied and generalized by many mathematicians in different areas of mathematics, such as inner product spaces, quadrature formulae, finite Fourier transforms, linear functionals, etc.

Along this work $\mathcal{H}$ denotes a (complex, separable) Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Let $(\mathbb{B}(\mathcal{H}), \| \cdot \|)$ be the C^*-algebra of all bounded linear operators acting on $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ with the uniform norm. We denote by Id the identity operator, and for any $A \in \mathbb{B}(\mathcal{H})$ we consider A^* its adjoint and $|A| = (A^*A)^{\frac{1}{2}}$ the absolute value of A. By $\mathbb{B}(\mathcal{H})^+$ we denote the cone of positive operators of $\mathbb{B}(\mathcal{H})$, i.e. $\mathbb{B}(\mathcal{H})^+ := \{T \in \mathbb{B}(\mathcal{H}) : \langle Th, h \rangle \geq 0 \, \forall h \in \mathcal{H}\}$. In the case when $\dim \mathcal{H} = n$, we identify $\mathbb{B}(\mathcal{H})$ with the full matrix algebra $\mathcal{M}_n$ of all $n \times n$ matrices with entries in the complex field $\mathbb{C}$. For each $T \in \mathbb{B}(\mathcal{H})$, we denote its spectrum by $\sigma(T)$, that is, $\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda Id \text{ is not invertible}\}$ and a complex number $\lambda \in \mathbb{C}$ is said to be in the approximate point spectrum of the operator T, and we denote by $\sigma_{ap}(T)$, if there is a sequence $\{x_n\}$ of unit vectors satisfying $(T - \lambda)x_n \to 0$.

For each operator T we consider

$$r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\} \qquad \text{spectral radius of } T,$$

$$W(T) = \sup\{\langle Th, h \rangle : \|h\| = 1\} \qquad \text{numerical range of } T$$

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and

$$w(T) = \sup\{|\lambda| : \lambda \in W(T)\} \qquad\qquad \text{numerical radius of } T.$$

Recall that for all $T \in \mathbb{B}(\mathcal{H})$, $r(T) \leq w(T) \leq \|T\| \leq 2w(T)$, $\sigma(T) \subseteq \overline{W(T)}$ and by the Toeplitz-Hausdorff's Theorem $W(T)$ is convex.

Renaud [8] gave a matrix analogue of Grüss inequality by replacing integrable functions by matrices and the integration by a trace function as follows: let $A, T \in \mathcal{M}_n$, suppose that $W(A)$ and $W(T)$ are contained in disks of radii R_A and R_T, respectively. Then for a positive semi-definite matrix P with $tr(P) = 1$ holds

$$|tr(PAT) - tr(PA)tr(PT)| \leq 4R_A R_T, \tag{1.2}$$

and if A and T are normal, the constant 4 can be replaced by 1. We can see can easily see that if $A = \alpha Id$ or $T = \beta Id$ with $\alpha, \beta \in \mathbb{C}$ then the left hand side is equal to zero. In the same article, Renaud propose the following open problem: to characterise $k(A, T)$, where

$$|tr(PAT) - tr(PA)tr(PT)| \leq k(A, T)R_A R_T, \tag{1.3}$$

with $1 \leq k(A, T) \leq 4$. In particular, whether it depends on A and T separately, i.e. whether we can write $k(A, T) = h(A)h(T)$, where $h(A), h(B)$ are suitably defined constants.

In this article we give a new proof and a refinement of Renuad's inequality and a postive answer to his open question.

2. Preliminaries

Let us begin with the notation and the necessary definitions.

The set of compact operators in $\mathcal{H}$ is denoted by $B_0(\mathcal{H})$. If $T \in B_0(\mathcal{H})$ we denote by $\{s_n(T)\}$ the sequence of singular values of T, i.e., the eigenvalues of $|T|$ (decreasingly ordered). The notion of unitary invariant norms can be defined also for operators on Hilbert spaces. A norm $|||.|||$ that satisfies the invariance property $|||UXV||| = |||X|||$. If $\dim R(T) = 1$, then $|||T||| = s_1(T)g(e_1) = g(e_1)\|T\|$. By convention, we assume that $g(e_1) = 1$. If $x, y \in \mathcal{H}$, then we denote $x \otimes y$ the rank one operator defined on $\mathcal{H}$ by $(x \otimes y)(z) = \langle z, y\rangle x$ then $\|x \otimes y\| = \|x\|\|y\| = |||x \otimes y|||$.

The most known examples of unitary invariant norms are the Schatten p-norms For $1 \leq p < \infty$, let

$$\|T\|_p^p = \sum_n s_n(T)^p = tr\,|X|^p\,,$$

and

$$B_p(\mathcal{H}) = \{T \in \mathcal{H} : \|T\|_p < \infty\},$$

called the $p-Schatten\,class$ of $\mathbb{B}(\mathcal{H})$. That is the subset of compact operators with singular values in l_p. The positive operators with trace 1 are called density operator (or states) and we denote this set by $\mathcal{S}(\mathcal{H})$. The ideal $B_2(\mathcal{H})$ is called the Hilbert-Schmidt class and it is a Hilbert space with the inner product $\langle S, T\rangle_2 = tr(ST^*)$. On the theory of norm ideals and their associated unitarily invariant norms, a reference for this subject is [4].

For $A, T \in \mathbb{B}(\mathcal{H})$ and $P \in \mathcal{S}(\mathcal{H})$ we introduce the following notation

$$V_P(A,T) = tr(PAT) - tr(PA)tr(PT).$$

In the particular case $T = A^*$ we get the variance of A respect to P. More precisely, Audenaert in [1] consider the following notion, given $A, P \in \mathcal{M}_n, P \geq 0, tr(P) = 1$ the variance of A respect to the matrix P

$$V_P(A) = tr(|A|^2 P) - |tr(AP)|^2 = V_P(A, A^*),$$

Note that $V_P(A - \lambda Id) = V_P(A)$. Futhermore, he showed that if $A \in \mathcal{M}_n$ then

$$\max\{tr(|A|^2 P) - |tr(AP)|^2 : P \in \mathcal{M}_n^+, tr(P) = 1\} = dist(A, \mathbb{C}Id)^2, \tag{2.1}$$

and the maximization over P on the left hand side can be restricted to density matrices of rank 1.

We will prove that the equality (2.1) holds in infinite dimensional context and as consequence of this fact we obtain the Renaud's result.

3. Distance formulas and Renaud's inequality

Let A and T linear bounded operators acting in $\mathcal{H}$, the vector-function $A - \lambda T$ is known as the pencil generated by A and T. Evidently there is at least one complex number λ_0 such that

$$\|A - \lambda_0 T\| = \inf_{\lambda \in \mathbb{C}} \|A - \lambda T\|.$$

The number λ_0 is unique if $0 \notin \sigma_{ap}(T)$ (or equivalently if $\inf\{\|Tx\| : \|x\| = 1\} > 0$). Different authors, following [9], called to this unique number as center of mass of A respect to T and we denote by $c(A,T)$ and when $T = Id$ we write $c(A)$. Following Paul, for $A, T \in \mathbb{B}(\mathcal{H})$ such that $0 \notin \sigma_{ap}(T)$ we consider

$$M_T(A) = \sup_{\|x\|=1} \left[\|Ax\|^2 - \frac{|\langle Ax, Tx\rangle|^2}{\langle Tx, Tx\rangle} \right]^{1/2} = \sup_{\|x\|=1} \left\| Ax - \frac{\langle Ax, Tx\rangle}{\langle Tx, Tx\rangle} Tx \right\|, \tag{3.1}$$

in [6], he proved that $M_T(A) = dist(A, \mathbb{C}T)$. The unique minimizer is characterized by the following conditions: there exists a sequence of unit vectores $\{x_n\}$ such that

$$\|(A - \lambda_0 T)x_n\| \to \|A - \lambda_0 T\| \qquad \text{and} \qquad \langle (A - \lambda_0 T)x_n, x_n\rangle \to 0.$$

Recently, S. Dragomir in [3] related the variance of A respect to P with the distance from A to the unidimensional subspace $\mathbb{C}Id$, more precisely he proved that for any $A \in \mathbb{B}(\mathcal{H})$ and $P \in \mathcal{S}(\mathcal{H})$ the following inequality holds

$$V_P(A)^{1/2} = [tr(|A|^2 P) - |tr(AP)|^2]^{1/2} \leq \min_{\lambda \in \mathbb{C}} \|A - \lambda Id\|.$$

Now, we present a new proof of this fact.

Proposition 3.1. *Let $A \in \mathbb{B}(\mathcal{H})$ and $P \in \mathcal{S}(\mathcal{H})$ then*

$$\begin{aligned} tr(|A|^2 P) - |tr(AP)|^2 &= \|AP^{1/2}\|_2^2 - |\langle AP^{1/2}, P^{1/2}\rangle_2|^2 \\ &= \|AP^{1/2} - \langle AP^{1/2}, P^{1/2}\rangle_2 P^{1/2}\|_2^2 \\ &= \min_{\lambda \in \mathbb{C}} \|AP^{1/2} - \lambda P^{1/2}\|_2^2. \end{aligned}$$

Proof. Is a simple consequence from following general statement for any Hilbert space $\mathcal{H}$: let $x, y \in \mathcal{H}$ with $y \neq 0$ then

$$\inf_{\lambda \in \mathbb{C}} \|x - \lambda y\|^2 = \frac{\|x\|^2\|y\|^2 - |\langle x, y\rangle|^2}{\|y\|^2}.$$

□

The following statement is an extensión of the Audenaert's formula to infinite dimension.

Theorem 3.2. *Let $A \in \mathbb{B}(\mathcal{H})$ then*

$$\sup\{[tr(|A|^2P) - |tr(AP)|^2]^{1/2} : P \in \mathcal{S}(\mathcal{H})\} = dist(A, \mathbb{C}Id). \tag{3.2}$$

Proof. We note that by [7] we get

$$\begin{aligned} dist(A, \mathbb{C}Id)^2 &= \sup_{\|x\|=1} \|Ax\|^2 - |\langle Ax, x\rangle|^2 \\ &\leq \sup\{tr(|A|^2P) - |tr(AP)|^2 : P \in \mathcal{S}(\mathcal{H})\} \\ &\leq dist(A, \mathbb{C}Id)^2. \end{aligned}$$

□

As consequence of the Audenaert's formula we get the following upper bound for $V_P(A, T)$.

Corollary 3.3. *Let $A, T \in \mathbb{B}(\mathcal{H})$ then*

$$|V_P(A, T)| \leq \sup_{P \in \mathcal{S}(\mathcal{H})} |tr(PAT) - tr(PA)tr(PT)| \leq dist(A, \mathbb{C}Id)dist(T, \mathbb{C}Id), \tag{3.3}$$

and in particular

$$|V_P(A, A^*)| \leq \sup_{P \in \mathcal{S}(\mathcal{H})} |tr(P|A|^2) - |tr(PA)|^2| \leq dist(A, \mathbb{C}Id)^2.$$

Proof. It is consequence of Theorem 3.2 and Theorem 6 in [3]. □

Remark 3.4. If we define $V_P : \mathbb{B}(\mathcal{H}) \times \mathbb{B}(\mathcal{H}) \to \mathbb{C}$,

$$V_P(A, T) := tr(PAT) - tr(PA)tr(PT).$$

Then V_P is a bilinear function and by (3.3) a continuous mapping with $\|V_P\| \leq 1$.

Now, we give a new proof and a refinement of (1.2).

Proposition 3.5. *Let $A, T \in \mathbb{B}(\mathcal{H})$ and we suppose that $W(A), W(T)$ are contained in closed disk $D(\lambda_0, R_A), D(\mu_0, R_T)$ respectively. Then for any $P \in \mathcal{S}(\mathcal{H})$*

$$\begin{aligned} |tr(PAT) - tr(PA)tr(PT)| &\leq \sup_{P \in \mathcal{S}(\mathcal{H})} |tr(PAT) - tr(PA)tr(PT)| \\ &\leq dist(A, \mathbb{C}Id)dist(T, \mathbb{C}Id) \\ &\leq \|A - \lambda_0 Id\|\|T - \mu_0 Id\| \\ &\leq 4w(A - \lambda_0 Id)w(T - \mu_0 Id) \\ &\leq 4R_A R_T. \end{aligned} \tag{3.4}$$

In particular, if A and T are normal operators, we have

$$\begin{aligned} |tr(PAT) - tr(PA)tr(PT)| &\leq \sup_{P\in\mathcal{S}(\mathcal{H})} |tr(PAT) - tr(PA)tr(PT)| \\ &\leq dist(A, \mathbb{C}Id)dist(T, \mathbb{C}Id) = r_A r_T, \end{aligned} \tag{3.5}$$

where r_S denotes the radius of the unique smallest disc containing $\sigma(S)$ for any $S \in \mathbb{B}(\mathcal{H})$.

Proof. The inequalities are consequence of Theorem 3.2 and (3.3). In the last inequality we use that $W(A - \lambda_0 Id) \subset D(0, R_A)$ and $W(T - \mu_0 Id) \subset D(0, R_T)$ respectively.

On the other hand, Björck and Thomée [2] have shown that for a normal operator A

$$dist(A, \mathbb{C}Id) = \sup_{\|x\|=1} (\|Ax\|^2 - |\langle Ax, x\rangle|^2)^{1/2} = r_A,$$

and this completes the proof. □

From (3.5), if we consider A is a positive invertible operator, $T = A^{-1}$ and $P = x \otimes x$ with $x \in \mathcal{H}$ with $\|x\| = 1$, then

$$\begin{aligned} |tr(PAT) - tr(PA)tr(PT)| &= |1 - \langle Ax, x\rangle\langle A^{-1}x, x\rangle| \\ &\leq dist(A, \mathbb{C}Id)dist(A^{-1}, \mathbb{C}Id) = r_A r_{A^{-1}}, \end{aligned}$$

i.e. we obtain the Kantorovich inequality for an operator A acting on an infinite dimensional Hilbert space $\mathcal{H}$ with $0 < m \leq A \leq M$.

Finally, in the following statement we give a positive answer at the Renuad's open question.

Theorem 3.6. *Let $A, T \in \mathbb{B}(\mathcal{H})$ with $A, T \notin \mathbb{C}Id$ and we suppose that $W(A), W(T)$ are contained in closed disk $D(\lambda_0, R_A), D(\mu_0, R_T)$ respectively. Thus for any $P \in \mathcal{S}(\mathcal{H})$ we get*

$$\begin{aligned} |tr(PAT) - tr(PA)tr(PT)| &\leq \sup_{P\in\mathcal{S}(\mathcal{H})} |tr(PAT) - tr(PA)tr(PT)| \\ &\leq dist(A, \mathbb{C}Id)dist(T, \mathbb{C}Id) \\ &\leq h(A)h(T)R_A R_T, \end{aligned} \tag{3.6}$$

where $h(A) = 1 + \frac{\|A-c(A)Id\|}{2w(A-\lambda_0 Id)}, h(T) = 1 + \frac{\|T-c(T)Id\|}{2w(T-\mu_0 Id)}$ and $1 \leq h(A)h(T) \leq 4$.

Proof. From (3.4) we have

$$\begin{aligned} |tr(PAT) - tr(PA)tr(PT)| &\leq \sup_{P\in\mathcal{S}(\mathcal{H})} |tr(PAT) - tr(PA)tr(PT)| \\ &\leq \|A - c(A)Id\|\|T - c(T)Id\|. \end{aligned} \tag{3.7}$$

On the other hand,

$$\begin{aligned} \|A-c(A)Id\| &\leq \frac{1}{2}\|A-c(A)Id\|+\frac{1}{2}\|A-\lambda_0 Id\| \\ &\leq \frac{1}{2}\|A-c(A)Id\|+w(A-\lambda_0 Id) \\ &= w(A-\lambda_0 Id)\left(1+\frac{\|A-c(A)Id\|}{2w(A-\lambda_0 Id)}\right) \\ &= h(A)w(A-\lambda_0 Id), \end{aligned} \tag{3.8}$$

where $h(A)\leq 2$ since $\|A-c(A)Id\|\leq\|A-\lambda_0 Id\|\leq 2w(A-\lambda_0 Id)$.

Thus, combining (3.7) and (3.8) we complete the proof. □

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