Best approximation, unitary groups and orbits of compact self-adjoint operators

Tamara Bottazzi* and Alejandro Varela[1]

*Sede Andina, Universidad Nacional de Río Negro;

[1] IAM-CONICET and Universidad Nacional de General Sarmiento.

Introduction

Let $\mathcal{H}$ be a separable Hilbert space, $\mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$ the anti-Hermitian bounded diagonals in some fixed orthonormal basis and $\mathcal{K}(\mathcal{H})$ the compact operators ($\mathcal{B}(\mathcal{H})$ is endowed by usual operator norm $\|\cdot\|$). We study the subgroup of unitary operators

$$\mathcal{U}_{k,d} = \{u \in \mathcal{U}(\mathcal{H}) : \exists\, D \in \mathcal{D}\left(\mathcal{B}\left(\mathcal{H}\right)\right)^{ah} / u - e^{D} \in \mathcal{K}(\mathcal{H})\}$$

in order to obtain a concrete description of short curves in unitary Fredholm orbits $\mathcal{O}_b = \{e^K b e^{-K} : K \in \mathcal{K}(\mathcal{H})^{ah}\}$ of a compact self-adjoint operator b with spectral multiplicity one. We consider the rectifiable distance on $\mathcal{O}_b$ defined as the infimum of curve lengths measured with the Finsler metric defined by means of the quotient space $\mathcal{K}(\mathcal{H})^{ah}/\mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})$.

Previous results

Minimality and best approximation in $\mathcal{B}(\mathcal{H})$

- Let $\mathcal{B} \subset \mathcal{A}$ von Neumann algebras. Then, for every $a \in \mathcal{A}^h$ there exists $b_0 \in \mathcal{B}^h$ such that
$$\text{dist}(a, \mathcal{B}) = \|a + b_0\|\,, [1].$$
- If $C \in \mathcal{K}(\mathcal{H})^{ah}$ has finite range, there exists $D_0 \in \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})$ such that
$$\text{dist}(C, \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})) = \|C + D_0\|\,, [2].$$

Geometry of operator algebras

- For every $c \in \mathcal{O}_b$ and $x \in T(\mathcal{O}_b)_c \cong \mathcal{K}(\mathcal{H})^{ah}/\mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})$ there exist a minimal lifting $Z_0 \in \mathcal{B}(\mathcal{H})^{ah}$ (in the quotient norm, not necessarily compact) such that $\gamma(t) = e^{tZ_0}\, c\, e^{-tZ_0}$ is a short curve on $\mathcal{O}_b$ in a certain interval, [1],[2].

Minimal Hermitian operator with non-compact diagonal 1

First case: Given any $\gamma \in (-1,1)$ and some $r \in \mathbb{R}_{>0}$

$$T_r = \begin{pmatrix} 0 & r\gamma & r\gamma^2 & r\gamma^3 & r\gamma^4 & \cdots \\ r\gamma & 0 & \gamma & \gamma^2 & \gamma^3 & \cdots \\ r\gamma^2 & \gamma & 0 & \gamma^2 & \gamma^3 & \cdots \\ r\gamma^3 & \gamma^2 & \gamma^2 & 0 & \gamma^3 & \cdots \\ r\gamma^4 & \gamma^3 & \gamma^3 & \gamma^3 & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \text{ then} \qquad (1)$$

- $tr(T_r T_r^*) < \infty$.
- $D_0 = \text{Diag}\left(\{d_k\}_{k\in\mathbb{N}}\right)$, such that
$$d_1 = 0, \text{ and for each } k \geq 2$$
$$d_k = -\sum_{j=0}^{k-3} \gamma^j - \frac{\gamma^k}{1-\gamma^2} = -\frac{1-\gamma^{k-2}}{1-\gamma} - \frac{\gamma^k}{1-\gamma^2}.$$
is the unique Diagonal such that $\|T_r + D_0\| = \text{dist}(T_r, \mathcal{D}(\mathcal{K}(\mathcal{H})))$
- $\lim_{k\to\infty} d_k = \dfrac{1}{\gamma - 1} \Rightarrow D_0$ is not compact.

Minimal length curves in $\mathcal{O}_b$: First case

- Case 1: Let $b = \text{Diag}\left(\{\lambda_n\}_{n\in\mathbb{N}}\right)$ with $\{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ such that $\lambda_n \neq \lambda_m$, $\forall\, n \neq m$. Consider T_r defined in (1) and the uniparametric curve
$$\beta(t) = e^{itT_r} b e^{-itZ_r}, t \in \mathbb{R}. \qquad (2)$$

Then,

1. $\beta(t) = e^{it\left(T_r + \frac{1}{1-\gamma} I\right)} b e^{-it\left(T_r + \frac{1}{1-\gamma} I\right)}$, i.e. $\beta \subset \mathcal{O}_b$.
2. $\begin{cases} \beta(0) = b, \\ \beta'(0) = iT_r b - biT_r \in (T\mathcal{O}_b)_b. \end{cases}$
3. Length $\left(\beta|_{[0,t_0]}\right) = |t_0|\,\|T_r\| = d(b, \beta(t_0))$ for each $t_0 \in \left[-\frac{\pi}{2\|T_r\|}, \frac{\pi}{2\|T_r\|}\right]$.

Minimal Hermitian operator with non-compact diagonal 2

Second case: Given any $\delta, \gamma \in (0,1)$ and some $r \in \mathbb{R}_{>0}$

$$Z_r = \begin{pmatrix} 0 & -r\delta & r\gamma & -r\delta^2 & r\gamma^2 & \cdots \\ -r\delta & 0 & \gamma & -\delta^2 & \gamma^2 & \cdots \\ r\gamma & \gamma & 0 & -\delta^2 & \gamma^2 & \cdots \\ -r\delta^2 & -\delta^2 & -\delta^2 & 0 & \gamma^2 & \cdots \\ r\gamma^2 & \gamma^2 & \gamma^2 & \gamma^2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \text{ then}$$

- $tr(Z_r Z_r^*) < \infty$.
- If $\gamma^2 \leq \delta \leq \gamma^{1/2}$ (see Fig. 1), there exists a unique bounded diagonal $D_1 = \text{Diag}\left(\{d'_n\}_{n\in\mathbb{N}}\right)$, such that
$$\|Z_r + D_1\| = \text{dist}(Z_r, \mathcal{D}(\mathcal{K}(\mathcal{H}))).$$

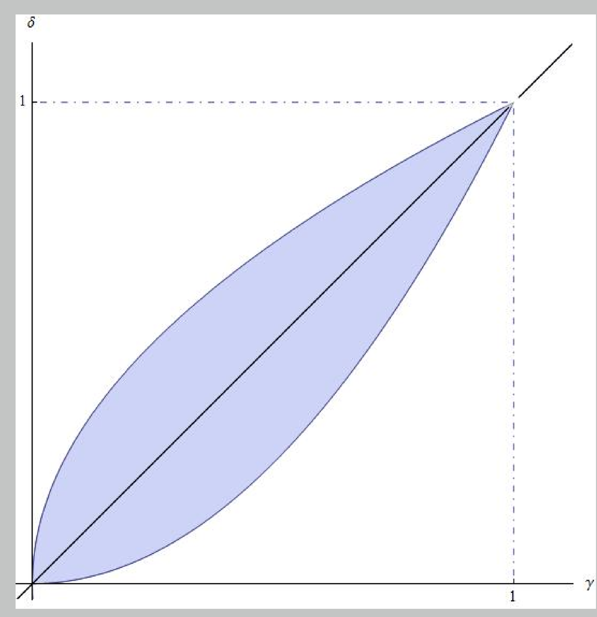

Figure 1:Region δ, γ

- D_1 fulfills
  - $d'_1 = 0$
  - $d'_{2k} = \left(\sum_{j=1}^{k-1} \delta^j\right) - \left(\sum_{j=1}^{k-1} \gamma^j\right) + \frac{\delta^{k+2}}{1-\delta^2} + \left(\frac{\gamma^2}{\delta}\right)^k \frac{1}{1-\gamma^2}$, for $k > 1$.
  - $d'_{2k-1} = \left(\sum_{j=1}^{k-1} \delta^j\right) - \left(\sum_{j=1}^{k-2} \gamma^j\right) - \frac{\gamma^{k+1}}{1-\gamma^2} - \left(\frac{\delta^2}{\gamma}\right)^k \frac{\gamma}{1-\delta^2}$, for $k > 1$.
  - $\lim_{k\to\infty} d'_{2k} \neq \lim_{k\to\infty} d'_{2k} \Rightarrow \nexists \lim_{n\to\infty} (D_1)_{nn}$.
- **Corollary**: there exists operators with "oscillant" non-compact minimal diagonal.

Minimal length curves in $\mathcal{O}_b$: Second case

- Case 2: Techniques used in Case 1 for minimal curves did not hold. We had to improve.
- Study of the unitary group $\mathcal{U}_{k,d}$. Properties, structure and relations with $\mathcal{U}_k$.
- Then, show that $\mathcal{O}_b^{\mathcal{U}_{k,d}} = \mathcal{O}_b$.
- $\gamma : \left[-\frac{\pi}{2\|Z_r\|}, \frac{\pi}{2\|Z_r\|}\right] \to \mathcal{O}_b$, $\gamma(t) = e^{tZ_r} b e^{-tZ_r}$ is a minimal length curve on $\mathcal{O}_b$ in the rectifiable distance.

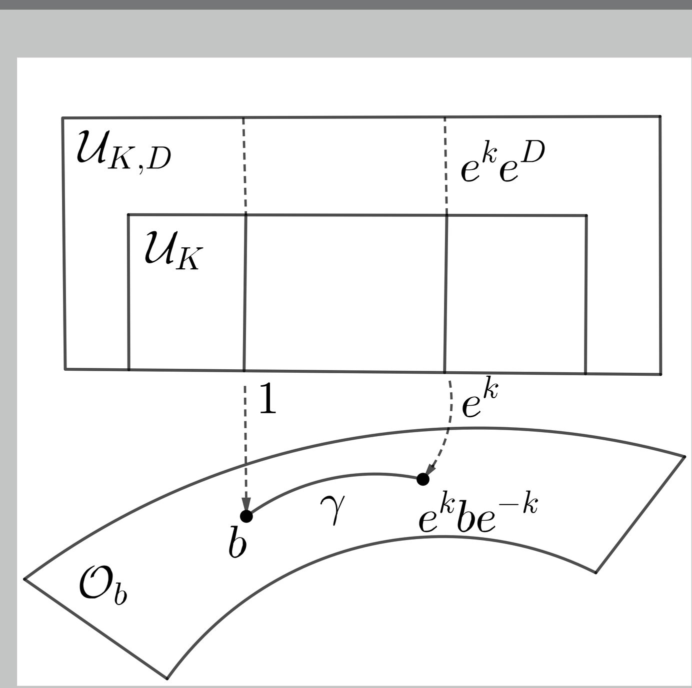

Figure 2:Orbit $\mathcal{O}_b$ and $\mathcal{U}_{k,d}$

References

[1] Carlos E. Durán, Luis E. Mata-Lorenzo, and Lázaro Recht. Metric geometry in homogeneous spaces of the unitary group of a C^*-algebra. I. Minimal curves. *Adv. Math.*, 184(2):342–366, 2004.

[2] Esteban Andruchow and Gabriel Larotonda. The rectifiable distance in the unitary Fredholm group. *Studia Math.*, 196(2):151–178, 2010.

[3] Tamara Bottazzi and Alejandro Varela. Unitary subgroups and orbits of compact self-adjoint operators. *Studia Math.*, 238(2):155–176, 2017.

[4] Tamara Bottazzi and Alejandro Varela. Minimal length curves in unitary orbits of a Hermitian compact operator. *Differential Geom. Appl.*, 45:1–22, 2016.

Acknowledgments

- To Open Arms organization for the grant provided to T. Bottazzi.
- To CONICET and UNRN for economical support.