Best approximation by diagonal operators in Schatten ideals

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Abstract

If $\mathcal{X}$ is the set of compact or $p-$Schatten operators over a complex Hilbert separable space $\mathcal{H}$, we study the existence and characterization properties of Hermitian $A \in \mathcal{X}$ such that

$$|||A||| \leq |||A + D|||, \text{ for all } D \in \mathcal{D}(\mathcal{X})$$

or equivalently

$$|||A||| = \min_{D \in \mathcal{D}(\mathcal{X})} |||A + D||| = \operatorname{dist}(A, \mathcal{D}(\mathcal{X})),$$

where $\mathcal{D}(\mathcal{X})$ is the subspace of diagonal operators of $\mathcal{X}$ in any prefixed basis of $\mathcal{H}$ and $||| \cdot |||$ is the usual operator norm in each $\mathcal{X}$. We use Birkhoff-James orthogonality as a tool to characterize and develop properties of these operators in each context. We also provide several illustrative examples.

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1. Introduction

The notion of orthogonality on an inner product space has been generalized to any normed space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ in several ways. One of the most

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studied is the so-called *Birkhoff–James* orthogonality [11, 22]: for $x, y \in \mathcal{X}$ it is said that x is *Birkhoff–James orthogonal* (B-J) to y, denoted by $x \perp_{BJ} y$, if and only if

$$\|x\| \leq \|x + \gamma y\| \tag{1.1}$$

for all $\gamma \in \mathbb{K}$. If $\mathcal{X}$ is an inner product space, then B-J orthogonality is equivalent to the usual orthogonality given by the inner product. It is also easy to see that B-J orthogonality is nondegenerate, is homogeneous, but it is neither symmetric nor additive. There are several works dedicated to the study of this type of orthogonality, we cite for example [11, 22, 23, 10, 9, 2, 25, 8, 32, 31, 35, 36] in chronological order.

In a similar way, for every closed subspace $\mathcal{B} \subseteq \mathcal{X}$ and $x \in \mathcal{X}$ we say x is Birkhoff–James orthogonal to $\mathcal{B}$ (noted by $x \perp_{BJ} \mathcal{B}$) if

$$\|x\| \leq \|x + b\|, \text{ for all } b \in \mathcal{B},$$

that is $\|x\| = \operatorname{dist}(x, \mathcal{B})$. This x is also called a minimal vector and observe that

$$x \perp_{BJ} \mathcal{B} \Leftrightarrow x \perp_{BJ} b \text{ for all } b \in \mathcal{B}. \tag{1.2}$$

Problems related with existence, unicity and characterization of minimal vectors in normed spaces were extensively studied in [3, 4, 5, 17, 20, 21]. Relative to the existence problem, if we consider $\mathcal{B} \subset \mathcal{A}$ von Neumann algebras and $a \in \mathcal{A}$, $a = a^*$, there always exists an element b_0 in $\mathcal{B}$ such that $\|a + b_0\| \leq \|a + b\|$, for all $b \in \mathcal{B}$ (see [17]). The element $a + b_0$ is minimal in the class $[a]$ of the quotient space $\mathcal{A}/\mathcal{B}$. However, in the case of $\mathcal{A} = \mathcal{K}(\mathcal{H})$, the C^*-algebra of compact operators over a complex Hilbert space $\mathcal{H}$ (which is not a von Neumann algebra), and $\mathcal{B} = \mathcal{D}(\mathcal{K}(\mathcal{H})) \subset \mathcal{K}(\mathcal{H})$, the C^*-subalgebra of diagonal operators respect some prefixed basis, there is not always exist a minimal Hermitian compact operator in each class on $[Z] = \{Z + D : \text{ such that } D \in \mathcal{D}(\mathcal{K}(\mathcal{H}))\}$. In [13] and [15] we exhibit examples of this fact. The existence of a best approximant for a compact Hermitian operator C is guaranteed for example when $\mathcal{H}$ is finite dimensional or C has finite rank [3].

We study in this paper the problem to find and characterize minimal Hermitian vectors in $\mathcal{X}$, where $\mathcal{X}$ can be the space of bounded linear, compact or $p-$Schatten operators, with $1 \leq p < \infty$, over $\mathcal{H}$. With this purpose, we use B-J orthogonality as a tool to characterize minimal Hermitian operators. In all cases, $\mathcal{B} = \mathcal{D}(\mathcal{X})$, that is the subspace of diagonal operators of $\mathcal{X}$ in any prefixed basis of $\mathcal{X}$. If a Hermitian operator $A \in \mathcal{X}$ is minimal, that is

$$|||A||| \leq |||A + D||| \text{ for all } D \in \mathcal{D}(\mathcal{X})$$

and $|||\cdot|||$ is the usual operator norm in each $\mathcal{X}$, then $\operatorname{Diag}(A)$ is the diagonal operator which minimizes the norm of $A - \operatorname{Diag}(A)$, or equivalently

$$|||A||| = \operatorname{dist}(A, \mathcal{D}(\mathcal{A})).$$

The problem about minimal operators is related with the study of minimal length curves of the orbit manifold of a Hermitian compact operator T by a particular unitary group, that is

$$\mathcal{O}_T = \{uTu^* : \ u \text{ is bounded, linear, unitary and } u - 1 \in \mathcal{X}\}.$$

The existence of a (not necessarily unique) minimal element A allows the description of minimal length curves of the manifold $\mathcal{O}_T$ with initial velocity $x = iAb - biA$ by the parametrization

$$\gamma(t) = e^{tiA}\ b\ e^{-tiA},\ t \in [-1, 1].$$

For a deeper discussion of this topic we refer the reader to [17, 3, 15].

We briefly describe the contents of this paper. Section 2 contains basic definitions, notation and some preliminary results. In section 3, we introduce the concept of minimality for bounded linear operators acting on $\mathcal{H}$ and we develop some general properties. In section 4, we present the concept of minimal operators in $p-$Schatten ideals for $1 < p < \infty$ and we relate it with Birkhoff-James orthogonality. In section 5 we focus on characterize minimal compact and trace class operators using Gateaux $\varphi-$derivatives. In the last section, we present and describe some particular results and cases for the minimality of compact Hermitian operators in the spectral norm.

2. Preliminaries

Let $(\mathcal{H}, \langle , \rangle)$ be a separable Hilbert space. As usual, $\mathcal{B}(\mathcal{H})$, $\mathcal{U}(\mathcal{H})$ and $\mathcal{K}(\mathcal{H})$ denote the sets of bounded, unitary and compact operators on $\mathcal{H}$. We denote with $\|\cdot\|$ the usual operator norm in $\mathcal{B}(\mathcal{H})$. The symbol I stands for the identity operator on $\mathcal{B}(\mathcal{H})$.

If an orthonormal basis $\{e_i\}_{i=1}^{\infty}$ of $\mathcal{H}$ is fixed we can consider matricial representations of each $A \in \mathcal{B}(\mathcal{H})$. More precisely, we regard an operator

$A \in \mathcal{B}(\mathcal{H})$ as an infinite matrix defined for each $i, j \in \mathbb{N}$ as $A_{ij} = \langle Ae_i, e_j \rangle$. In this sense, ith-row of A and the jth-column are the vectors in ℓ^2 given by and $f_j(A) = (A_{i1}, A_{i2}, ...)$ and $c_j(A) = (A_{1j}, A_{2j}, ...)$, respectively.

If $\mathcal{A}$ is any subspace of $\mathcal{B}(\mathcal{H})$, we denote with $\mathcal{D}(\mathcal{A})$ the set of diagonal operators with respect to the prefixed basis of $\mathcal{H}$, that is

$$\mathcal{D}(\mathcal{A}) = \{A \in \mathcal{A} : \ \langle Ae_i, e_j \rangle = 0 \ , \quad \text{for all } i \neq j\} \, .$$

We define the operator $\mathrm{Diag} : \mathcal{B}(\mathcal{H}) \to \mathcal{D}(\mathcal{B}(\mathcal{H}))$, which essentially takes the main diagonal of an operator A (i.e the elements of the form $\langle Ae_i, e_i \rangle_{i\in\mathbb{N}}$) and builds a diagonal operator in the prefixed basis of $\mathcal{H}$. For a given sequence $\{d_n\}_{n\in\mathbb{N}}$ we denote with $\mathrm{Diag}\big((d_n)_{n\in\mathbb{N}}\big)$ the diagonal (infinite) matrix with $(d_n)_{n\in\mathbb{N}}$ in its diagonal and 0 elsewhere.

Given a subspace $\mathcal{S}$ of $\mathcal{H}$, we denote as $P_{\mathcal{S}}$ the orthogonal projection onto $\mathcal{S}$. For every subset $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$, we use the superscript ^{h} to note the subset of Hermitian elements of $\mathcal{A}$. A Hermitian element $A \in \mathcal{B}(\mathcal{H})$ is called positive if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathcal{H}$ and it is denoted by $A \geq 0$. For an operator $A \in \mathcal{B}(\mathcal{H})$ we use $\ker(A)$ to denote the kernel of A and it can be defined the modulus of A as $|A| = (A^*A)^{\frac{1}{2}}$.

For every compact operator $A \in \mathcal{K}(\mathcal{H})$, let $s_1(A), s_2(A), \cdots$ be the singular values of A, i.e. the eigenvalues of $|A|$ in decreasing order ($s_i(A) = \lambda_i(|A|)$, for each $i \in \mathbb{N}$) and repeated according to multiplicity. For $p > 0$, let

$$\|A\|_p = \left(\sum_{i=1}^{\infty} s_i(A)^p \right)^{\frac{1}{p}} = (\mathrm{tr}|A|^p)^{\frac{1}{p}} \, , \tag{2.1}$$

where $\mathrm{tr}(\cdot)$ is the trace functional, i.e.

$$\mathrm{tr}(A) = \sum_{j=1}^{\infty} \langle Ae_j, e_j \rangle. \tag{2.2}$$

Note that this coincides with the usual definition of the trace if $\mathcal{H}$ is finite-dimensional. We observe that the series (2.2) converges absolutely and it is independent from the choice of basis. Equality (2.1) defines for $1 \leq p < \infty$ a norm on the ideal

$$\mathcal{B}_p(\mathcal{H}) = \{A \in \mathcal{K}(\mathcal{H}) : \|A\|_p < \infty\},$$

called the *p-Schatten class*.

Note that if $\mathcal{H} = \mathbb{C}^n$, for every $n \in \mathbb{N}$, $\mathcal{B}_p(\mathbb{C}^n)$ is the space of square $n \times n$ complex matrices endowed with the $\|\cdot\|_p$ Schatten norm.

We summarize some of the most important properties of p-Schatten operators in the following theorem.

Theorem 2.1. *Let $1 \leq p < \infty$, then*

1. $\mathcal{B}_p(\mathcal{H}) \subseteq \mathcal{K}(\mathcal{H})$.
2. $\mathcal{B}_p(\mathcal{H})$ *is an operator ideal in* $\mathcal{B}(\mathcal{H})$ *and a Banach space with the* $\|\cdot\|_p$ *norm.*
3. *for every* $A \in \mathcal{B}_p(\mathcal{H})$ *and* $T \in \mathcal{B}(\mathcal{H})$ *we have the following inequalities:*
$$\|A\| \leq \|A\|_p = \|A^*\|_p \text{ and } \|TA\|_p \leq \|T\|\|A\|_p.$$
4. *Hölder inequality: for every* $A \in \mathcal{B}_p(\mathcal{H})$ *and* $T \in \mathcal{B}_q(\mathcal{H})$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$
$$|\mathrm{tr}(AT)| \leq \|AT\|_1 \leq \|A\|_p \|T\|_q$$

For $1 < p < \infty$, $(\mathcal{B}_p(\mathcal{H}), \|.\|_p)$ is a uniformly convex space as a consequence of the classical McCarthy-Clarkson inequality (see [30], Theorem 2.7). The ideal $\mathcal{B}_1(\mathcal{H})$ is called the *trace class*. It is not reflexive and, in particular, is not a uniformly convex space, because it contains $\mathcal{D}(\mathcal{B}_1(\mathcal{H}))$, which is isomorphic to l_1. Other relevant ideal is $\mathcal{B}_2(\mathcal{H})$, the *Hilbert-Schmidt class*, and it is a Hilbert space with the inner product $\langle A, B\rangle_{HS} := \mathrm{tr}(B^*A)$.

Let $E_{ij} = e_i \otimes e_j$, with $\{e_i\}_{i=1}^{\infty}$ the fixed orthonormal basis of $\mathcal{H}$. Observe that every E_{ij} is a rank one operator for all $i, j \in \mathbb{N}$ and any $A \in \mathcal{K}(\mathcal{H})$ can be written as $A = \sum_{i,j=1}^{\infty} a_{ij} E_{ij}$, with $a_{ij} = \langle Ae_i, e_j\rangle$. The set $\{E_{ij}\}_{i,j=1}^{\infty}$ is an orthonormal countable basis of $\mathcal{B}_2(\mathcal{H})$, since

$$\langle E_{ij}, E_{kl}\rangle_{HS} = \mathrm{tr}(E_{kl}^* E_{ij}) = \begin{cases} 1 & \text{if } i = k \text{ and } j = l \\ 0 & \text{in any other case.} \end{cases}$$

The Schatten p-norms and the operator norm are special examples of unitarily invariant norms, i.e. $|||UXV||| = |||X|||$, for every pair of unitary operators U, V. On the theory of norm ideals and their associated unitarily invariant norms, a reference for this subject is [19].

In a normed space $\mathcal{X}$, $x \in \mathcal{X}$ is a smooth point if there is a unique hyperplane supporting the open ball $B(0, \|x\|)$ at x. We say that $\mathcal{X}$ is a smooth space if all its points in the unit sphere are smooth points. For geometric and topological properties of smooth points in Banach spaces we refer to [16] and references therein.

3. Generalities of minimal operators in $\mathcal{X}^h/\mathcal{D}(\mathcal{X})^h$

for every $\mathcal{X}$ closed subspace of $\mathcal{B}(\mathcal{H})$ and $A \in \mathcal{X}^h$ we say that A is minimal in the norm $|||\cdot|||$ of $\mathcal{X}$ if and only if

$$|||A||| \leq |||A + D||| \text{ for all } D \in \mathcal{D}(\mathcal{X}^h), \tag{3.1}$$

or equivalently, $\text{dist}_{|||\cdot|||}(A, \mathcal{D}(\mathcal{X}^h)) = |||A|||$. This is equivalent to say that the norm of A is the quotient norm of the class $\{A + D : \; D \in \mathcal{D}(\mathcal{X}^h)\}$ in the quotient space $\mathcal{X}^h/\mathcal{D}(\mathcal{X}^h)$. In this case, we say that the diagonal of A, $\text{Diag}(A)$, is minimizant or is the best approximant of A in $\mathcal{D}(\mathcal{X}^h)$. In case of existence, the best Hermitian (or real) diagonal aproximation of an operator may not be unique. In this sense, another equivalent way to consider the minimality problem is, given an operator $A_0 \in \mathcal{X}^h$, find $D_0 \in \mathcal{D}(\mathcal{X}^h)$ such that

$$|||A_0 + D_0||| \leq |||A_0 + D||| \text{ for all } D \in \mathcal{D}(\mathcal{X}^h). \tag{3.2}$$

In (3.2), A_0 can be taken with zero diagonal. We will use both formulations (3.1) or (3.2) whenever is convenient.

Remark 3.1. for every operator $X \in \mathcal{X}$, if $|||\cdot|||$ is self-adjoint norm (i.e. $|||X^*||| = |||X|||$)

$$|||Re(X)||| = \frac{1}{2}|||(X + X^*)||| \leq |||X|||$$

it follows that $A \in \mathcal{X}^h$ is minimal if and only if

$$|||A||| \leq |||A + D|||, \text{ for all } \; D \in \mathcal{D}(\mathcal{X}). \tag{3.3}$$

The next two Propositions are closely related with the Hahn-Banach theorem for Banach spaces and they link the ideal spaces $\mathcal{B}_p(\mathcal{H})$ and $\mathcal{B}(\mathcal{H})_q$ with $\frac{1}{p} + \frac{1}{q} = 1$. Both results are generalizations of the Banach Duality formula found in [13] and we include proofs for the sake of completeness. To simplify, here we use the notation $\mathcal{B}_\infty(\mathcal{H}) = \mathcal{K}(\mathcal{H})$.

Proposition 3.2. *Let $A \in \mathcal{B}_p(\mathcal{H})$, $1 \leq p \leq \infty$ and consider the set*

$$\mathcal{N}_q = \left\{ Y \in \mathcal{B}_q(\mathcal{H})^h : \ \|Y\|_q = 1, \ tr(YD) = 0 \text{ for all } D \in \mathcal{D}(\mathcal{B}_p(\mathcal{H})) \right\},$$

with $\frac{1}{p} + \frac{1}{q} = 1$. Then, there exists $Y_0 \in \mathcal{N}$ such that

$$\|[A]\|_{\mathcal{B}_p(\mathcal{H})^h/\mathcal{D}(\mathcal{B}_p(\mathcal{H}))^h} = \inf_{D \in \mathcal{D}(\mathcal{B}_p(\mathcal{H}))} \|C + D\|_p = tr(Y_0 A). \tag{3.4}$$

For $1 < p < \infty$ this Y_0 is unique and has the form

$$Y_0 = \frac{|A|^{p-1}U^*}{\||A|^{p-1}U^*\|_q},$$

where U is the partial isometry of the polar decomposition of A.

Proof. The existence is an immediate consequence from the Hahn-Banach theorem that since $\mathcal{D}(\mathcal{B}_p(\mathcal{H}))^h$ is a closed subspace of $\mathcal{B}_p(\mathcal{H})^h$ for all $1 \leq p \leq \infty$. Then there exists a functional $\rho : \mathcal{K}(\mathcal{H}) \to \mathbb{R}$ such that $\|\rho\| = 1$, $\rho(D) = 0$, for all $D \in \mathcal{D}(\mathcal{B}_p(\mathcal{H})^h)$, and

$$\rho(A) = \inf_{D \in \mathcal{D}(\mathcal{B}_p(\mathcal{H})^h)} \|A + D\|_p = \operatorname{dist}(A, \mathcal{D}(\mathcal{B}_p(\mathcal{H})^h)).$$

But, since any functional ρ can be written as $\rho(.) = \operatorname{tr}(Y_0.)$, with $Y_0 \in \mathcal{B}_q(\mathcal{H})$, the result follows.

On the other hand, for $1 < p < \infty$ and using that $\frac{1}{p} + \frac{1}{q} = 1$

$$\left\||A|^{p-1}\right\|_q^q = \operatorname{tr}\left(|A|^{p-1}\right)^q = \operatorname{tr}|A|^p = \|A\|_p^p$$

and so $\left\||A|^{p-1}\right\|_q = \|A\|_p^{p/q} = \|A\|_p^{p-1}$, this implies that $|A|^{p-1} \in \mathcal{B}_q(\mathcal{H})$. The operator $Y_0 \in N_q$ from (3.5) can be defined as

$$Y_0 = \frac{|A|^{p-1}U^*}{\||A|^{p-1}U^*\|_q}$$

and the support functional is unique since $\mathcal{B}_p(\mathcal{H})$ with $1 < p < \infty$ is a uniformly convex space and every $A \in \mathcal{B}_p(\mathcal{H})$ is a smooth point (see [1]). $\square$

Proposition 3.3 (Banach Duality Formula)**.** *Let $A \in \mathcal{B}_p(\mathcal{H})^h$ with $1 \leq p \leq \infty$ and $q \in \mathbb{R}$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\inf_{D \in \mathcal{D}(\mathcal{B}_p(\mathcal{H})^h)} \|A + D\|_p = \max_{Y \in \mathcal{N}_q} |tr(AY)|, \tag{3.5}$$

Proof. Let $A \in \mathcal{B}_p(\mathcal{H})^h$ with p, q as in the hypothesis. By Proposition 3.2, there exists $Y_0 \in \mathcal{N}_q$ such that

$$\inf_{D \in \mathcal{D}(\mathcal{B}_p(\mathcal{H})^h)} \|A + D\|_p = \operatorname{tr}(Y_0 A).$$

Then

$$\begin{aligned} \inf_{D \in \mathcal{D}(\mathcal{B}_p(\mathcal{H})^h)} \|A + D\|_p &= \operatorname{tr}(Y_0 A) \leq \max_{Y \in \mathcal{N}_q} |\operatorname{tr}(AY)| \max_{Y \in \mathcal{N}_q} |\operatorname{tr}((A + D)Y)| \\ &\leq \|Y\|_q \|A + D\|_p = \|A + D\|_p, \end{aligned}$$

for every $D \in \mathcal{D}(\mathcal{B}_p(\mathcal{H})^h)$, where the last inequality is due to item (6) of Theorem 2.1. Therefore, the equality (3.5) can be proved as a consequence of this fact. □

Propositions 3.2 (existence condition) and 3.3 can be generalized to any closed subspace $\mathcal{B}$ of $\mathcal{B}_p(\mathcal{H})$, not only for $\mathcal{D}(\mathcal{B}_p(\mathcal{H}))$, with the same arguments.

Proposition 3.4. *If A is a minimal operator in $\mathcal{B}_p(\mathcal{H})^h$, $1 < p \leq \infty$, and $A \geq 0$ then $A = 0$. That is, any nonzero minimal Hermitian operator cannot be positive semidefinite.*

Proof. $A \geq 0$ implies that $A = U|A| = |A|$ and $U = I$. We separate the proof in different cases.

- Case $p = \infty$: if A is minimal and positive then by the balanced spectrum property (Prop. 6 in [13]) $\lambda_{max} = -\lambda_{min} = 0$. Hence, $A = 0$.
- Case $1 < p < \infty$: $A \geq 0$ and minimal implies by Theorem 4.5

  $$tr(|A|^{p-1}U^*) = tr(A^{p-1}) = \sum_{i=1}^{n} \lambda_i(A^{p-1}) = 0.$$

  By continuous funcional calculus, for all $1 < p < \infty$

  $$\lambda_i(A^{p-1}) = \lambda_i(A)^{p-1}$$

  with $\lambda_i(A) \geq 0$ for all i. Then $\lambda_i(A) = 0$ for all i and $A = 0$.

$\square$

In the case $p = 1$, there are minimal positive operators also in a finite dimensional context. We can see an example in Remark 4.10.

4. BJ-orthogonality in $\mathcal{B}_p(\mathcal{H})$ and minimality of Hermitian operators

Let $\mathcal{B}_p(\mathcal{H})$ be a p-Schatten ideal with $p > 0$. Using (1.1) the Birkhoff–James orthogonality for every $A, B \in \mathcal{B}_p(\mathcal{H})$ is

$$A \perp_{BJ}^{p} B \text{ if and only if } \|A\|_p \leq \|A + \gamma B\|_p \text{ for all } \gamma \in \mathbb{C}.$$

Let $\mathcal{D}(\mathcal{B}_p(\mathcal{H}))$ be the closed subspace of diagonal operators of $\mathcal{B}_p(\mathcal{H})$, that is

$$\mathcal{D}(\mathcal{B}_p(\mathcal{H})) = \left\{ D \in \mathcal{D}(\mathcal{K}(\mathcal{H})) : \sum_{i=1}^{\infty} |\langle De_i, e_i \rangle|^p < \infty \right\}.$$

By (1.2), given $A \in \mathcal{B}_p(\mathcal{H})$,

$$A \perp_{BJ} \mathcal{D}(\mathcal{B}_p(\mathcal{H})) \Leftrightarrow A \perp_{BJ}^{p} D \text{ for all } D \in \mathcal{D}(\mathcal{B}_p(\mathcal{H})). \tag{4.1}$$

We focus in particular when $p \geq 1$, where $\|\cdot\|_p$ is a norm.

According with (3.1) we say that $A \in \mathcal{B}_p(\mathcal{H})^h$ is minimal in the $p-$Schatten norm if and only if

$$\|A\|_p \leq \|A + D\|_p, \text{ for all } D \in \mathcal{D}(\mathcal{B}_p(\mathcal{H}))^h.$$

The operator A is minimal in the class $[A] = \{A + D : D \in \mathcal{D}(\mathcal{B}_p(\mathcal{H}))^h\}$ of the quotient space $\mathcal{B}_p(\mathcal{H})^h / \mathcal{D}(\mathcal{B}_p(\mathcal{H})^h)$.

Applying Remark 3.1, we can combine minimality with BJ- orthogonality as follows: given $A \in \mathcal{B}_p(\mathcal{H})^h$

$$A \text{ is minimal if and only if } A \perp_{BJ} D(\mathcal{B}_p(\mathcal{H})). \tag{4.2}$$

Remark 4.1. Since every $\mathcal{B}_p(\mathcal{H})$ with $1 < p < \infty$ is a uniformly convex Banach space and $\mathcal{D}(\mathcal{B}_p(\mathcal{H}))$ is a proper closed vector subspace, there always exists a unique minimal element A in its class, that is $\|A\| = \mathrm{dist}_p(A, \mathcal{D}(\mathcal{B}_p(\mathcal{H}))^h)$ (Lemma 4 in [18]).

4.1. Minimality in $\mathcal{B}_2(\mathcal{H})$

Here we consider the Hilbert-Schmidt class endowed with the inner product $\langle A, B\rangle_{HS} = tr(B^*A)$, $A, B \in \mathcal{B}_2(\mathcal{H})$ and its induced 2-norm, that is $\|A\|_2 = \sqrt{tr(A^*A)}$. Then, we have the following minimality theorem.

Theorem 4.2. *Let $A \in \mathcal{B}_2(\mathcal{H})$, then the following conditions are equivalent:*

1. $\|A\|_2 = \min\limits_{D\in\mathcal{D}(\mathcal{B}_2(\mathcal{H}))} \|A + D\|_2$.
2. $\operatorname{Diag}(A) = 0$.

Proof. From the theory of approximation in Hilbert spaces and since $\mathcal{D}(\mathcal{B}_2(\mathcal{H}))$ is a closed subspace of $\mathcal{B}_2(\mathcal{H})$, we obtain that the problem

$$\min_{D\in\mathcal{D}(\mathcal{B}_2(\mathcal{H}))} \|A + D\|_2 \tag{4.3}$$

has unique diagonal solution of the form

$$\begin{pmatrix} \langle A, E_{11}\rangle_{HS} & 0 & \cdots & \cdots & \cdots \\ 0 & \langle A, E_{22}\rangle_{HS} & \ddots & \cdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ \vdots & \cdots & 0 & \langle A, E_{nn}\rangle_{HS} & \\ \vdots & \cdots & \cdots & \cdots & \ddots \end{pmatrix} = -\operatorname{Diag}(A), \tag{4.4}$$

which is provided by the normal equations in the minimum least squares problem ($A + D \perp E_{ii}$ for all $i \in \mathbb{N}$, and $\{E_{ii}\}_{i\in\mathbb{N}}$ is a basis for $\mathcal{D}(\mathcal{B}_2(\mathcal{H}))$). Thus,

$$\min_{D\in\mathcal{D}(\mathcal{B}_2(\mathcal{H}))} \|A + D\|_2 = \|A - \operatorname{Diag}(A)\|_2$$

and

$$\begin{aligned} \operatorname{dist}_2(A, \mathcal{D}(\mathcal{B}_2(\mathcal{H})))^2 &= \|A - \operatorname{Diag}(A)\|_2^2 \\ &= tr\left(A^*A - A^*\operatorname{Diag}(A) - \operatorname{Diag}(A)^*A + \operatorname{Diag}(A)^*\operatorname{Diag}(A)\right) \\ &= \|A\|_2^2 - \|\operatorname{Diag}(A)\|_2^2. \end{aligned}$$

If $\|A\|_2 = \operatorname{dist}_2(A, \mathcal{D}(\mathcal{B}_2(\mathcal{H})))$, by the unicity of the solution of (4.3) $A = A - \operatorname{Diag}(A)$, then $\operatorname{Diag}(A) = 0$. On the other hand, if $\operatorname{Diag}(A) = 0$, then

$$\operatorname{dist}_2(A, \mathcal{D}(\mathcal{B}_2(\mathcal{H})))^2 = \|A\|_2^2 - \|\operatorname{Diag}(A)\|_2^2 = \|A\|_2^2.$$

□

Corollary 4.3. *Let $A \in \mathcal{B}_2(\mathcal{H})^h$, then A is a minimal operator if and only if* $\operatorname{Diag}(A) = 0$.

4.2. $\mathcal{B}_p(\mathcal{H})$ ideals as semi-inner product spaces and minimality

Lumer [29] and Giles [18] noticed that in any normed space $(\mathcal{X}, \|\cdot\|)$ it can be can construct a semi-inner product, i.e., a mapping $[\cdot,\cdot] : \mathcal{X} \times \mathcal{X} \to \mathbb{K}$ such that

(1) $[x, x] = \|x\|^2$,

(2) $[\alpha x + \beta y, z] = \alpha[x, z] + \beta[y, z]$,

(3) $[x, \gamma y] = \overline{\gamma}[x, y]$,

(4) $|[x, y]|^2 \leq \|x\|^2\|y\|^2$,

for all $x, y, z \in \mathcal{X}$ and all $\alpha, \beta, \gamma \in \mathbb{K}$. It is well known that in a normed space there exists exactly one semi-inner product if and only if the space is smooth (i.e., there is a unique support hyperplane at each point of the unit surface). If $\mathcal{X}$ is an inner product space, the only semi inner product on $\mathcal{X}$ is the inner product itself. More details can be found in [29, 18].

Proposition 4.4 ([12])**.** *Let $1 < p < \infty$ and we define for every $A, B \in \mathcal{B}_p(\mathcal{H})$*

$$[B, A] = \|A\|_p^{2-p}\mathrm{tr}\left(|A|^{p-1}U^*B\right), \tag{4.5}$$

where $U|A|$ is the polar decomposition of A. Then, $\left(\mathcal{B}_p(\mathcal{H}), [\cdot,\cdot]\right)$ is a continuous semi-inner product space (in the Lumer sense) and the following statements are equivalent:

(i) $A \perp_{BJ}^p B$.

(ii) $[B, A] = 0$.

Observe that this semi-inner product does not fulfill the conjugate property, since in general $[B, A] \neq \overline{[A, B]}$.

Theorem 4.5. *Let $1 < p < \infty$, $\{e_i\}_{i\in\mathbb{N}}$ be the fixed basis of $\mathcal{H}$ and $A \in \mathcal{B}_p(\mathcal{H})$ with the polar decomposition $A = U|A|$. The following conditions are equivalent:*

1. $A \perp_{BJ}^p \mathcal{D}(\mathcal{B}_p(\mathcal{H}))$.
2. $[D, A] = 0$ *for all* $D \in \mathcal{D}(\mathcal{B}_p(\mathcal{H}))$.

3. $tr(|A|^{p-1}U^*D) = 0$ *for all* $D \in \mathcal{D}(\mathcal{B}_p(\mathcal{H}))$.
4. $tr(|A|^{p-1}U^*E_{ii}) = 0$, *for all* $E_{ii} = \mathrm{Diag}(e_i)$, $i \in \mathbb{N}$.
5. $(|A|^{p-1}U^*)_{ii} = 0$ *for all* $i \in \mathbb{N}$.
6. $(U|A|^{p-1})_{ii} = 0$ *for all* $i \in \mathbb{N}$.

Proof. The equivalence betwenn items (1), (2) and (3) are direct consequence of Prop. 4.4.

(3)⇒(4) is trivial since each $E_{ii} \in \mathcal{D}(\mathcal{B}_p(\mathcal{H}))$ for all $i \in \mathbb{N}$.

(4)⇒(3) occurs since any $D \in \mathcal{D}(\mathcal{B}_p(\mathcal{H}))$ can be written as $D = \sum_{i=1}^{\infty} d_i E_{ii}$, with $d_i = \langle De_i, e_i \rangle$ such that $\sum_{i=1}^{\infty} |d_i|^p < \infty$ and

$$tr(|A|^{p-1}U^*D) = \lim_{N\to\infty} tr\left(|A|^{p-1}U^*\left(\sum_{i=1}^{N} d_i E_{ii}\right)\right)$$

$$= \lim_{N\to\infty} \sum_{i=1}^{N} d_i tr\left(|A|^{p-1}U^*E_{ii}\right) = 0.$$

(4)⇒(5): each E_{ii} can be written as $e_i \otimes e_i$. Then

$$0 = tr(|A|^{p-1}U^*E_{ii}) = \sum_{j=1}^{\infty} \left\langle |A|^{p-1}U^*E_{ii}e_j, e_j \right\rangle = \left\langle |A|^{p-1}U^*e_i, e_i \right\rangle$$

$$= (|A|^{p-1}U^*)_{ii}$$

for all $i \in \mathbb{N}$. The converse (5)⇒(4) is obvious.

(5)⇔(6): Using continuous funcional calculus for $|A|$ and $f(z) = z^{p-1}$

$$\left(|A|^{p-1}\right)^* = f(|A|)^* = \overline{f}(|A|) = (|A|^*)^{p-1} = |A|^{p-1}.$$

Then $|A|^{p-1}$ is Hermitian and for all $i \in \mathbb{N}$

$$0 = \overline{(|A|^{p-1}U^*)_{ii}} = (|A|^{p-1}U^*)^*_{ii} = (U|A|^{p-1})_{ii}.$$

□

Corollary 4.6. *Let* $A \in \mathcal{B}_p(\mathcal{H})^h$, *with* $2 \le p < \infty$. *Then*

1. *A is a minimal operator in the* $p-$*Schatten norm if and only if*

$$\mathrm{Diag}(A|A|^{p-2}) = \mathrm{Diag}(|A|^{p-2}A) = 0.$$

2. *For every even integer p, A is minimal if and only if* $\operatorname{Diag}(A^{p-1}) = 0$.
3. *If A is minimal in the $p-$Schatten norm, then*

$$\sum_{i=1}^{\infty} \operatorname{sgn}(\lambda_i(A))|\lambda_i(A)|^{p-1} = 0. \tag{4.6}$$

Proof. (1): Let $A \in \mathcal{B}_p(\mathcal{H})^h$, $p \geq 2$, be a minimal operator. If $U|A|$ is the polar decomposition of A, then by Theorem 4.5

$$(U|A|^{p-1})_{ii} = (U|A||A|^{p-2})_{ii} = (A|A|^{p-2})_{ii} = 0 \text{ for all } i \in \mathbb{N}.$$

Item (2) is due to Lemma 4.3 in [4], since A is minimal if and only if $tr(A^{p-1}D) = 0$ for all $D \in \mathcal{D}(\mathcal{B}_p(\mathcal{H})^h)$. In addition, observe that $f(t) = t^{p-1}$ is well defined for compact Hermitian (non positive) operators only if $p-1$ is not an even number.

(3): condition $\operatorname{Diag}(|A|^{p-2}A) = 0$ in particular implies that $tr(|A|^{p-2}A) = 0$. Also, by hypothesis A is diagonalizable, then there exists V a unitary operator in $\mathcal{B}(\mathcal{H})$ such that $A = V^* \operatorname{Diag}(\lambda(A))V$. Therefore,

$$\begin{aligned} tr\left(|A|^{p-2}A\right) &= tr\left(V^*|\operatorname{Diag}(\lambda(A))|^{p-2}\operatorname{Diag}(\lambda(A))V\right) \\ &= tr\left(|\operatorname{Diag}(\lambda(A))|^{p-2}\operatorname{Diag}(\lambda(A))\right) \\ &= \sum_{i=1}^{\infty} \operatorname{sgn}(\lambda_i(A))|\lambda_i(A)|^{p-1} = 0. \end{aligned}$$

□

Condition (3) in Corollary 4.6 gives a necessary condition over the eigenvalues of an operator to be minimal (not sufficient), anagously to the balanced spectrum property for minimal Hermitian compact operators in the spectral norm.

Remark 4.7 (Some examples of minimal operators).

1. $\mathcal{B}_p(\mathbb{C}^2)$ and $p \geq 2$: by (4.6) any non zero minimal matrix in the $p-$ Schatten norm $A = \begin{pmatrix} a & c \\ \bar{c} & d \end{pmatrix}$ fulfills that $\lambda_2(A) = -\lambda_1(A)$ for all $p \geq 2$. Then,

$$\|A\|_p = (|\lambda_1(A)|^p + |-\lambda_1(A)|^p)^{1/p} = 2^{1/p}|\lambda_1(A)|$$

and the characteristic polynomial of A is $p(\lambda) = \lambda^2 - (|c|^2 + a^2)$ since has no lineal term (i.e, $d = -a$). Therefore,

$$\|A\|_p = 2^{1/p}(|c|^2 + a^2)^{1/2}$$

is minimum when $a = 0$ and A has zero diagonal.

2. $\mathcal{B}_4(\mathbb{C}^3)$: if $A = \begin{pmatrix} a & 1 & 0 \\ 1 & b & 1 \\ 0 & 1 & c \end{pmatrix}$ is the minimal matrix in its class, then $\mathrm{Diag}(A) = 0$. Indeed, the matrix $A_0 = A - \mathrm{Diag}(A)$ fulfills that $A_0^3 = 2A_0$ and clearly $\mathrm{Diag}(A_0^3) = 0$, then A_0 is minimal.
   By a similar argument and for every separable $\mathcal{H}$, every tridiagonal Hermitian operator in $\mathcal{B}_4(\mathcal{H})$ with zero diagonal is minimal in its class.
3. However, not every Hermitian operator with zero diagonal is minimal in $\mathcal{B}_p(\mathcal{H})$. For example, when $p = 4$, $\mathcal{H} = \mathbb{C}^3$, and $A = \begin{pmatrix} 0 & a & b \\ \overline{a} & 0 & c \\ \overline{b} & \overline{c} & 0 \end{pmatrix}$, with $a, b, c \neq 0$ can not be the minimal matrix in its class, since
$$\mathrm{Diag}(A^3) = (ac\overline{b} + \overline{ac}b)I \neq 0.$$
   But also observe that if any of a, b or c is 0, then A is a minimal matrix.
4. $\mathcal{B}_p(\mathcal{H})^h$, $1 < p < \infty$: Let $A \in \mathcal{B}_p(\mathcal{H})^h$ be a block-diagonal operator, that is
$$A = \begin{pmatrix} A_1 & 0 & 0 & \cdots \\ 0 & A_2 & 0 & \cdots \\ 0 & 0 & A_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \begin{array}{c} \text{with } A_i = P_{S_i} A P_{S_i} \in \mathcal{B}_p(\mathcal{H})^h \\ \text{and } \oplus_{i=1}^{\infty} S_i = \mathcal{H} \ \forall\, i \in \mathbb{N}. \end{array}$$
   Then, there exists a unique $D_i \in \mathcal{D}(\mathcal{B}_p(\mathcal{H})^h)$ such that $A_i + D_i$ is minimal in the $p-$Schatten norm for all $i \in \mathbb{N}$. Therefore the block diagonal
$$A + D_0 = \begin{pmatrix} A_1 + D_1 & 0 & 0 & \cdots \\ 0 & A_2 + D_2 & 0 & \cdots \\ 0 & 0 & A_3 + D_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$
   is a minimal operator since
$$\begin{aligned} \|A + D_0\|_p^p &= \sum_{i=1}^{\infty} \|A_i + D_i\|_p^p \qquad (4.7) \\ &\leq \sum_{i=1}^{\infty} \|A_i + D_i'\|_p^p = \left\| \begin{pmatrix} A_1 + D_1' & 0 & 0 & \cdots \\ 0 & A_2 + D_2' & 0 & \cdots \\ 0 & 0 & A_3 + D_3' & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \right\|_p^p \end{aligned}$$

for all $D_i' \in \mathcal{D}(\mathcal{B}_p(\mathcal{H}))$. The first equality in (4.7) is a property of pinching operators [7].

Remark 4.8 (Considerations about Theorem 4.5 and Corollary 4.6). With the same notation and hypothesis of the mentioned results:

1. These results generalize for all $p \in (1, \infty)$ Lemma 4.3 in [4] about minimal (lifting) operators in $i\mathcal{B}_p(\mathcal{H})^h$, the subspace of anti-Hermitian operators in $\mathcal{B}_p(\mathcal{H})$.
2. Observe that for $p = 2$, $A \perp_{BJ} \mathcal{D}(\mathcal{B}_p(\mathcal{H}))$ if and only if $A_{ii} = (A^*)_{ii} = (|A|U^*)_{ii} = 0$ for all $i \in \mathbb{N}$, which equivalent to the characterization of minimal operators found in Theorem 4.2 using least squares.
3. The minimality condition in $\mathcal{B}_p(\mathcal{H})^h$
$$[E_{ii}, A] = 0 \text{ for all } i \in \mathbb{N}$$
is equivalent to the normal equations for the solution of the least squares problem in a non-Hilbert space context.
4. Any minimal operator A in $\mathcal{B}_p(\mathcal{H})$, with $p \in (1, \infty)$, fulfills $[D, A] = 0$ for all $D \in \mathcal{D}(\mathcal{B}_p(\mathcal{H}))$. But in general $[A, D] \neq 0$, since
$$[A, D] = 0 \text{ for all } D \in \mathcal{D}(\mathcal{B}_p(\mathcal{H})) \Leftrightarrow \|D\|_p^{2-p} \mathrm{tr}\left(|D|^{p-1} V^* A\right) = 0$$
$$\sum_{k=1}^{\infty} |d_k|^{p-1} e^{i\theta_k} A_{ii} = 0 \text{ for all } \{d_n\}_{n \in \mathbb{N}} \in \ell_p \Leftrightarrow \mathrm{Diag}(A) = 0$$
Here $V = \mathrm{Diag}\left(\{e^{-i\theta_k}\}_{k \in \mathbb{N}}\right)$ is the diagonal unitary operator of the polar decomposition of D. For example when $p = 2$ or item (3) in Remark 4.7, the minimal operator A satisfies that $\mathrm{Diag}(A) = 0$, then $[D, A] = 0$ and $[A, D] = 0$.

The case $p = 1$ is treated more generally in the next section, but when $\mathcal{H} = \mathbb{C}^n$, Theorem 2.1 in [10] can be used in order to have the following result.

Proposition 4.9. *for every matrix A with polar decomposition $A = U|A|$ in $\mathcal{B}_1(\mathbb{C}^n)$ such that $tr(U^*D) = 0$ for all $D \in \mathcal{D}(M_n(\mathbb{C}))$, then A is a minimal matrix in the trace norm. The converse is true if A is also invertible.*

Condition $tr(U^*D) = 0$ for all $D \in \mathcal{D}(M_n(\mathbb{C}))$ implies that the partial isometry U of the polar decomposition of A has null diagonal ($U \in \mathcal{N}_1$).

Remark 4.10 (Minimal matrices in the trace norm). 1. Minimal matrices in the 1−Schatten norm may be not unique, for example, if $\mathcal{H} = \mathbb{C}^2$ and U is a unitary matrix of 2×2, then

$$tr(U^*D) = 0 \text{ for all } D \in \mathcal{D}(\mathbb{C}^{2\times 2}) \Leftrightarrow\ U = \begin{pmatrix} 0 & e^{i\theta} \\ e^{i\beta} & 0 \end{pmatrix},\ \theta, \beta \in [0, 2\pi).$$

Therefore, any $A = A^*$ minimal (non-diagonal) has an unitary U for its polar decomposition as before and $U^*A = AU \geq 0$. If $A = \begin{pmatrix} a & c \\ \overline{c} & d \end{pmatrix}$ with $a, d \in \mathbb{R}$ and $c \in \mathbb{C}_{\neq 0}$, then

$$\begin{pmatrix} 0 & e^{-i\theta} \\ e^{-i\beta} & 0 \end{pmatrix} \begin{pmatrix} a & c \\ \overline{c} & d \end{pmatrix} = \begin{pmatrix} a & c \\ \overline{c} & d \end{pmatrix} \begin{pmatrix} 0 & e^{i\beta} \\ e^{i\theta} & 0 \end{pmatrix} = |A|$$

$$\Leftrightarrow \begin{pmatrix} \overline{c}e^{-i\theta} & de^{-i\theta} \\ ae^{-i\beta} & ce^{-i\beta} \end{pmatrix} = \begin{pmatrix} ce^{i\theta} & ae^{i\beta} \\ de^{i\theta} & \overline{c}e^{i\beta} \end{pmatrix} = |A|.$$

Simple calculations show that $\theta = -\arg(c) = -\beta$, $a = d$,

$$A = \begin{pmatrix} a & c \\ \overline{c} & a \end{pmatrix} \text{ with } |a| \leq |c| \text{ and } \|A\|_1 = |a + |c|| + |a - |c|| = 2|c|.$$

Observe that we characterized all minimal Hermitian matrices in $\mathbb{C}^{2\times 2}$ for the 1−Schatten norm in terms of c and a. Moreover, condition $|a| \leq |c|$ indicates that there is not unicity of the minimizant diagonal (in fact, there are infinite, all scalar multiples of the identity, with a scalar with modulus less or equal than $|c|$). For example, if $c = 1$, every $A = \begin{pmatrix} a & 1 \\ 1 & a \end{pmatrix}$ with $|a| \leq 1$ is a minimal matrix with the trace norm $|a + 1| + |a - 1| = 2$.
Also, there are minimal matrices that do not have necessary a zero diagonal. But in all cases, the minimizant diagonal is a scalar multiple of the identity, therefore

$$\text{dist}_1(A, \mathcal{D}(\mathbb{C}^{2\times 2})) = \text{dist}_1(A, \mathbb{C}I).$$

2. Any matrix $A = \begin{pmatrix} a & d & 0 \\ \overline{d} & b & 0 \\ 0 & 0 & c \end{pmatrix}$ with $d \neq 0$ fixed is a block diagonal matrix and

$$\|A\|_1 = \left\| \begin{pmatrix} a & d \\ \overline{d} & b \end{pmatrix} \right\|_1 + |c|$$

and it is minimal in the $1-$Schatten norm if and only if $c=0$, $a=b$ and $|a|\leq|d|$. One can check that a partial isometry of the polar decomposition that satisfies Prop. 4.9 is

$$U=\begin{pmatrix}0 & e^{i\theta_1} & 0\\ e^{i\theta_2} & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\ \text{with } \theta_1=-\theta_2,$$

it is not unitary ($ker(U)=ker(A)=\text{span}\{e_3\}$) and it can not be extended to an unitary with zero diagonal. There are infinite minimal matrices in the class of A and we observe that there are minimizant diagonals which are not a scalar multiple of the identity matrix.

3. Recall that the partial isometry of the polar decomposition of any matrix may not be unique but a minimal matrix must have any partial isometry of its polar decomposition with zero diagonal. Indeed, consider the case $\mathcal{B}_1(\mathbb{C}^3)$ and the matrix $A=\begin{pmatrix}a & 1 & 1\\ 1 & b & 1\\ 1 & 1 & c\end{pmatrix}$, with $a,b,c\in\mathbb{R}$ to be determined. Its polar decomposition has a unitary operator and every unitary U to fulfill the zero diagonal condition is given by

$$\begin{pmatrix}0 & e^{i\theta_1} & 0\\ 0 & 0 & e^{i\theta_2}\\ e^{i\theta_3} & 0 & 0\end{pmatrix}\ \text{or}\ \begin{pmatrix}0 & 0 & e^{i\theta_1}\\ e^{i\theta_2} & 0 & 0\\ 0 & e^{i\theta_3} & 0\end{pmatrix} \tag{4.8}$$

Then, $A=\begin{pmatrix}a & 1 & 1\\ 1 & b & 1\\ 1 & 1 & c\end{pmatrix}$ is a minimal matrix in $\mathcal{B}_1(\mathbb{C}^3)$ if $A=U|A|$ with U as in (4.8). Without loss of generality, we choose the first option. Then,

$$AU=U^*|A|\geq 0 \Leftrightarrow \begin{pmatrix}e^{i\theta_3} & ae^{i\theta_1} & e^{i\theta_2}\\ e^{i\theta_3} & e^{i\theta_1} & be^{i\theta_2}\\ ce^{i\theta_3} & e^{i\theta_1} & e^{i\theta_2}\end{pmatrix}=\begin{pmatrix}e^{-i\theta_3} & e^{-i\theta_3} & ce^{-i\theta_3}\\ ae^{-i\theta_1} & e^{-i\theta_1} & e^{-i\theta_1}\\ e^{-i\theta_2} & be^{-i\theta_2} & e^{-i\theta_2}\end{pmatrix}\geq 0$$

$$\Leftrightarrow \theta_k=0 \text{ for } k=1,2,3 \text{ and } a=b=c=1.$$

Therefore, $A=\begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix}=(1,1,1)\otimes(1,1,1)$ is a minimal matrix in its class and we observe that is semi-definite positive of rank one. The unitary U chosen to the condition was a permutation of the identity (in fact, $A = I|A|$ but I does not fulfill the minimality condition). In this case

$$\inf_{D\in\mathcal{D}(\mathcal{B}_1(\mathbb{C}^3))} \|A+D\|_1 = \|A\|_1 = 3$$

and, similar as the previous example,

$$\operatorname{dist}_1(A, \mathcal{D}(\mathbb{C}^{3\times 3})) = \operatorname{dist}_1(A, \mathbb{C}I).$$

5. Gateaux derivative and minimality of Hermitian operators in $\mathcal{B}_1(\mathcal{H})$ and $\mathcal{K}(\mathcal{H})$

In this section we focus on the study of the minimal Hermitian operators on the particular cases $\mathcal{B}_1(\mathcal{H})$ and $\mathcal{K}(\mathcal{H})$, which are not included in the study made previously. We follow central ideas from [27], [26], [1] and [24]. There are also more recent related work (see for instance [37] and [34]).

Definition 5.1. Let $(\mathcal{X}, \|\cdot\|)$ be an arbitrary Banach space. The $\varphi-$Gateaux derivative of the norm at the point x in the $y-$direction is

$$D_{\varphi,x}(y) = \lim_{t\to 0^+} \frac{\|x+te^{i\varphi}y\| - \|x\|}{t} \tag{5.1}$$

The case $\varphi = 0$ corresponds to the usual Gateaux derivative of the norm at the point x. In this case, the norm $\|\cdot\|$ is Gateaux differentiable at a nonzero $x \in \mathcal{X}$ if

$$\lim_{t\to 0^+} \frac{\|x+ty\| - \|x\|}{t} = ReD_x(y) \text{ for all } y \in \mathcal{X}, \tag{5.2}$$

where D_x is the unique functional in $\mathcal{X}^*$ (dual space of $\mathcal{X}$) such that $D_x(x) = \|x\|$ and $\|D_x\| = 1$. This condition is equivalent to say that x is a smooth point on the sphere $\mathcal{S}(0, \|x\|) \subset \mathcal{X}$. Relative to smooth points we collect the following facts.

Remark 5.2 (Smooth points).
1. For $1 < p < \infty$, every $A \in \mathcal{B}_p(\mathcal{H})$, $A \neq 0$, is a smooth point, since $\mathcal{B}_p(\mathcal{H})$ is a uniformly convex space. In this case

$$D_A(B) = tr\left(\frac{|A|^{p-1}UB^*}{\|A\|_p^{p-1}}\right) = \frac{1}{\|A\|_p}[B, A],$$

where $A = U|A|$ is the polar decomposition of A and $[\cdot,\cdot]$ is the semi-inner product defined in (4.5).

2. For $p = 1$, $A \in \mathcal{B}_1(\mathcal{H})$, $A \neq 0$, is a smooth point if and only if A or A^* are one-to-one. In the case that A is one-to-one,
$$D_A(B) = tr\,(UB^*)\,, \tag{5.3}$$
where $A = U|A|$ is the polar decomposition of A.
3. In $\mathcal{K}(\mathcal{H})$, $A \in \mathcal{K}(\mathcal{H})$ is a smooth point if and only if there exists a unique norm 1 vector v (up to multiplication by constants of modulus one) such that $\|A\| = \|Av\|$. in this case
$$D_A(B) = tr\left(\frac{v \otimes Av}{\|A\|}B\right) = \left\langle Bv, \frac{Av}{\|A\|}\right\rangle. \tag{5.4}$$

The next result is similar to Prop. 4.9 in an infinite dimensional context.

Proposition 5.3. *Let $A \in \mathcal{B}_1(\mathcal{H})$ be a smooth point. Then, the following statements are equivalent:*

1. $A \perp_{BJ} \mathcal{D}(\mathcal{B}_1(\mathcal{H}))$.
2. $D_A(D) = tr\,(U^*D) = 0$ *for all* $D \in \mathcal{D}(\mathcal{B}_1(\mathcal{H}))$ *if* A *is one-to-one (or* $tr\,(UD) = 0$ *if* A^* *is one-to-one).*
3. $U^*_{ii} = 0$ *for all* $i \in \mathbb{N}$, *i.e.* $\mathrm{Diag}(U^*) = 0$ *(or* $\mathrm{Diag}(U) = 0$ *if* A^* *is one-to-one).*

Here U is the partial isometry of the polar decomposition of A.

Proof. Without loss of generality, we assume A is one-to-one.

(1)$\Leftrightarrow$ (2) By [Lemma 1, [27]] $A {\perp_{BJ}}^1 D \Leftrightarrow D_A(D) = 0$. Using this fact for all $D \in \mathcal{D}(\mathcal{B}_1(\mathcal{H}))$ and formula in (5.3) we obtain the desired result.

(2)$\Leftrightarrow$ (3) This equivalence follows by the same argument as in the proof of (4)$\Leftrightarrow$ (5) from Theorem 4.5. □

Corollary 5.4. *Let $A \in \mathcal{B}_1(\mathcal{H})^h$ be a smooth point. Then, the following statements are equivalent:*

1. *A is minimal in $\mathcal{B}_1(\mathcal{H})$.*
2. $tr\,(UD) = 0$ *for all* $D \in \mathcal{D}(\mathcal{B}_1(\mathcal{H}))$.
3. $\mathrm{Diag}(U) = 0$.

Proof. Supose A is minimal in $\mathcal{B}_1(\mathcal{H})$, then by (4.2) it is equivalent to $A \perp_{BJ} \mathcal{D}(\mathcal{B}_1(\mathcal{H}))$. By Proposition 5.3,
$$D_A(D) = tr(U^*D) = tr(UD) = 0 \text{ for all } D \in \mathcal{D}(\mathcal{B}_1(\mathcal{H}))$$
since A is Hermitian and a smooth point. □

We obtain an analogous of Prop. 5.3 for smooth points in $\mathcal{K}(\mathcal{H})$.

Proposition 5.5. *Let $A \in \mathcal{K}(\mathcal{H})$ a smooth point and $v \in \mathcal{H}$ be the unique (up to multiplication by scalars of modulus one) unitary vector such that $\|A\| = \|Av\|$. Then, the following statements are equivalent:*

1. $A \perp_{BJ} \mathcal{D}(\mathcal{K}(\mathcal{H}))$.
2. $D_A(D) = tr\left(\frac{v \otimes Av}{\|A\|} D\right) = \left\langle Dv, \frac{Av}{\|A\|}\right\rangle = 0$ *for all* $D \in \mathcal{D}(\mathcal{K}(\mathcal{H}))$ *(in fact, for every* $D \in \mathcal{D}(\mathcal{B}(\mathcal{H}))$*).*

Proof. By Lemma 1 in [27] $A \perp_{BJ} D \Leftrightarrow D_A(D) = 0$. Using this fact for all $D \in \mathcal{D}(\mathcal{K}(\mathcal{H}))$ and formula in (5.4) we obtain the desired result.

□

By the mentioned balanced spectrum property, every $A \in \mathcal{K}(\mathcal{H})^h$ minimal fulfills that $\pm\|A\|$ is in the (discrete) spectrum. Then, there exist v, w linearly independent unitary eigenvectors of $\|A\|$ and $-\|A\|$, respectively. Therefore, a minimal operator A in $\mathcal{K}(\mathcal{H})^h$ can not be smooth. For non smooth points in a normed space $\mathcal{X}$, Keckic proved in [Theorem 1.4,[26]] that for every pair $x, y \in \mathcal{X}$

$$x \perp_{BJ} y \ \Leftrightarrow \inf_{\varphi} D_{\varphi,x}(y) \geq 0. \tag{5.5}$$

Also, he found explicit formulas for the Gateaux derivative for non smooth points on $\mathcal{B}_1(\mathcal{H})$ and $\mathcal{K}(\mathcal{H})$ ((5.6) and (5.8), resp.)

Now we characterize minimal Hermitian operators in $\mathcal{B}_1(\mathcal{H})$ and $\mathcal{K}(\mathcal{H})$, respectively.

Theorem 5.6. *Let $A \in \mathcal{B}_1(\mathcal{H})^h$. Then, the following statements are equivalent:*

1. *A is minimal.*
2. $|tr\,(U^*D)| \leq \|PDP\|_1$ *for all* $D \in \mathcal{D}(\mathcal{B}_1(\mathcal{H}))^h$*, where* $A = U|A|$ *is the polar decomposition of* A *and* $P = P_{ker(A)}$.

Proof. $A \in \mathcal{B}_1(\mathcal{H})^h$ is minimal if and only if $A \perp_{BJ} D$ for all $D \in \mathcal{D}(\mathcal{B}_1(\mathcal{H}))^h$. Then, by (5.5) it is equivalent to

$$\inf_{\varphi} D_{\varphi,A}(D) \geq 0 \text{ for all } D \in \mathcal{D}(\mathcal{B}_1(\mathcal{H}))^h.$$

The explicit formula found in [26] of the Gateaux derivative for $A, B \in \mathcal{B}_1(\mathcal{H})$ is

$$\lim_{t\to 0^+} \frac{\|A+tD\|_1 - \|A\|_1}{t} = Re\,(tr(U^*D)) + \|QDP\|_1, \tag{5.6}$$

where $A = U|A|$ is the polar decomposition of A, $P = P_{ker(A)}$ and $Q = P_{ker(A^*)}$. Replacing by A Hermitian and $B = e^{i\varphi}D$

$$\inf_{\varphi} D_{\varphi,A}(D) = \inf_{\varphi} \left(Re\left(e^{i\varphi} tr(U^*D)\right)\right) + \|PDP\|_1.$$

Therefore, $\inf_\varphi D_{\varphi,A}(D) \geq 0$ if and only if $|tr\,(U^*D)| \leq \|PDP\|_1$ for all $D \in \mathcal{D}(\mathcal{B}_1(\mathcal{H}))^h$, since

$$\inf_{\varphi} \left(Re\left(e^{i\varphi} tr(U^*D)\right)\right) = Re\left(e^{-i\arg(tr(U^*D))} tr(U^*D)\right) = Re\left(|tr(U^*D)|\right)$$

$\square$

Example 5.7. Suppose $A \in \mathcal{B}_1(\mathcal{H})^h$ and $\mathcal{S}$ is a finite dimension subspace of $\mathcal{H}$ such that A can be written by block notation as $A = \begin{array}{cc} \begin{bmatrix} A_{\mathcal{S}} & 0 \\ 0 & 0 \end{bmatrix} & \begin{array}{c} \mathcal{S} \\ \mathcal{S}^\perp \end{array} \end{array}$, where $A_{\mathcal{S}} = P_{\mathcal{S}} A P_{\mathcal{S}}$ is the compression of A by $\mathcal{S}$. Then, simple computations show that $P = P_{Ker(A)} = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$ and a partial isometry U for the polar decomposition is $U = \begin{bmatrix} U_{\mathcal{S}} & 0 \\ 0 & 0 \end{bmatrix}$. Therefore, by Theorem 5.6 A is minimal if and only if $|tr(U^*D)| \leq \|PDP\|_1$ for all $D \in \mathcal{D}(\mathcal{B}_1(\mathcal{H}))$. In particular, if $\mathcal{S} = \text{span}\{e_i : \ 1 \leq i \leq n\}$ then

$$|U_{ii}| = |tr(U^*E_{ii})| \leq \|PE_{ii}P\|_1 = \left\{ \begin{array}{ll} 0 & \text{if } i \leq n \\ 1 & \text{if } i \geq n \end{array} \right. . \text{ Thus } \text{Diag}(U) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

Theorem 5.8. *Let $A \in \mathcal{K}(\mathcal{H})^h$. Then, the following statements are equivalent:*

1. *A is minimal.*
2. $\inf\limits_{0\leq\varphi<2\pi} \max\limits_{\substack{v\in M_A,\\ \|v\|=1}} Re\left(e^{i\varphi}\langle U^*Dv, v\rangle\right) \geq 0$ *for all $D \in \mathcal{D}(\mathcal{K}(\mathcal{H})^h)$, where $A = U|A|$ is the polar decomposition of A and M_A is the subspace where the operator A attains its norm ($M_A \neq \emptyset$).*
3. *There exists $v \in M_A$ such that $\langle Dv, Av\rangle = 0$ for all $D \in \mathcal{D}(\mathcal{K}(\mathcal{H})^h)$.*

4. *There exists $v \in M_A$ such that for each $i \in \mathbb{N}$*

$$v_i = 0 \text{ or } (Av)_i = f_i(A)^t v = 0.$$

5. *There exists $v \in M_A$ such that $\langle Dv, Av\rangle = 0$ for all $D \in \mathcal{D}(\mathcal{B}(\mathcal{H})^h)$.*

Proof. (1)$\Leftrightarrow$(2): Analogously to the case $\mathcal{B}_1(\mathcal{H})$, an operator $A \in \mathcal{K}(\mathcal{H})^h$ is minimal if and only if

$$\inf_{\varphi} D_{\varphi,A}(D) \geq 0 \text{ for all } D \in \mathcal{D}(\mathcal{K}(\mathcal{H})^h).$$

In this case, the explicit formula found in [26] of the Gateaux derivative for $A, B \in \mathcal{K}(\mathcal{H})$ is

$$\lim_{t\to 0^+} \frac{\|A+tB\| - \|A\|}{t} = \max_{\substack{v\in\Phi,\\ \|v\|=1}} Re \left\langle U^* Bv, v\right\rangle . \tag{5.7}$$

where $A = U|A|$ is the polar decomposition of A and Φ is the characteristic subspace of $|A|$ respect to its eigenvalue s_1. Replacing by A Hermitian and $B = e^{i\varphi}D$

$$\inf_{\varphi} D_{\varphi,A}(D) = \inf_{0\leq\varphi<2\pi} \max_{\substack{v\in M_A,\\ \|v\|=1}} Re\left(e^{i\varphi}\left\langle U^* Dv, v\right\rangle\right).$$

Therefore, A is minimal if and only if

$$\inf_{0\leq\varphi<2\pi} \max_{\substack{v\in M_A,\\ \|v\|=1}} Re\left(e^{i\varphi}\left\langle U^* Dv, v\right\rangle\right) \geq 0 \text{ for all } D \in \mathcal{D}(\mathcal{K}(\mathcal{H})^h)$$

.

Equivalence (2)$\Leftrightarrow$(3) is due to Corollary 2.8 in [26] for $A \in \mathcal{K}(\mathcal{H})^h$ fixed and every $B = D \in \mathcal{D}(\mathcal{K}(\mathcal{H})^h)$. Also, (1) $\Leftrightarrow$ (3) can be obtained by Corollary 2.2.1 in [33].

For item (2) observe that the set M_A cannot be empty since $\mathcal{H}$ is reflexive and $A \in \mathcal{K}(\mathcal{H})$.

(3) $\Leftrightarrow$ (4): statement (3) is equivalent to say that there exists $v \in M_A$ such that

$$\begin{aligned} \langle E_{ii}v, Av\rangle &= \langle v_i e_i, Av\rangle = v_i\overline{(Av)_i} = 0 \text{ for all } i \in \mathbb{N} \\ &\Leftrightarrow\ v_i = 0 \ \vee\ (Av)_i = f_i(A)^t v = 0 \text{ for all } i \in \mathbb{N}. \end{aligned}$$

(3) $\Leftrightarrow$ (5): it is evident since condition (3) is equivalent to

$$\langle E_{ii}v, Av\rangle = 0 \text{ for all } i \in \mathbb{N}$$

and every $D \in \mathcal{D}(\mathcal{B}(\mathcal{H}))$ can be written as $\sum_{i=1}^{\infty} d_i E_{ii}$ with $d_i \in \mathbb{C}$, $|d_i| \le M$ for some $M > 0$. □

Observe that equivalence between statements (1) and (5) in Theorem 5.8 gives a characterization for minimal Hermitian operators with bounded non-compact diagonal. It is an improvement from Lemma 6.1 in [14]. Aditionally, note that the vector of condition (3) in the same theorem cannot be an eigenvector of the minimal operator A and fulfills that

$$\|(A + D)v\|^2 = \|Av\|^2 + \|Dv\|^2$$

for all $D \in \mathcal{D}(\mathcal{B}(\mathcal{H}))$.

Remark 5.9 (Minimal operators with non compact diagonal). There are operators $A \in \mathcal{K}(\mathcal{H})^h$ such that $\operatorname{dist}(A, \mathcal{D}(\mathcal{K}(\mathcal{H}))^h)$ is attained by bounded diagonals that are not compact. One relevant example is the following: let Z_r be the operator defined matricially as

$$Z_r = \begin{pmatrix} 0 & r\gamma & r\gamma^2 & r\gamma^3 & \cdots \\ r\gamma & d_2 & \gamma & \gamma^2 & \cdots \\ r\gamma^2 & \gamma & d_3 & \gamma^2 & \cdots \\ r\gamma^3 & \gamma^2 & \gamma^2 & d_4 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \text{ with } \begin{cases} \gamma \in (0,1). \\ d_n = -\frac{1-\gamma^{n-2}}{1-\gamma} - \frac{\gamma^n}{1-\gamma^2}, n \ge 2. \\ r = \frac{\left\|Z_r^{[1]}\right\|}{\sqrt{\frac{\gamma^2}{1-\gamma^2}}}, \end{cases}$$

where $Z_r^{[1]}$ is the operator defined by the matrix of Z_r with zeros in the first column and row. Then, in [13] we proved that Z_r is a minimal operator with $Z_r - \operatorname{Diag}(Z_r) \in \mathcal{B}_2(\mathcal{H})^h \subset \mathcal{K}(\mathcal{H})^h$ and $\operatorname{Diag}(Z_r)$ is the (uniquely determined) diagonal minimizant, but it is not compact, since $\lim_{n\to\infty} d_n \neq 0$. Moreover, $\|Z_r\| = \|c_1(Z_r)\| = \|Z_r e_1\|$. Also,

$$\langle De_1, Z_r e_1\rangle = \langle D_{11}e_1, c_1(Z_r)\rangle = 0 \text{ for all } D \in \mathcal{D}(\mathcal{B}(\mathcal{H})^h),$$

which means that Z_r fulfills items (3) and (4) of Theorem 5.8 for $v = e_1$.

Curiously, we observe that by Corollary 4.3 $Z_r - \operatorname{Diag}(Z_r)$ is indeed a minimal operator in the quotient space $\mathcal{B}_2(\mathcal{H})^h/\mathcal{D}(\mathcal{B}_2(\mathcal{H}))^h$.

6. Cases of minimal Hermitian operators in $\mathcal{K}(\mathcal{H})$

Let $M_n(\mathbb{C})$ be the vector space of complex $n \times n$ matrices. In this context, we say $M \in M_n(\mathbb{C})^h$ is minimal in the spectral norm if

$$\|M\| \leq \|M + D\|$$

for all D real diagonal $n \times n$ matrix ($D \in \mathcal{D}(M_n(\mathbb{C})^h)$). Several characterizations and studies of geometric consequences of minimal Hermitian matrices in the spectral norm can be found in [5], [6] and [28]. We continue with some examples of minimal Hermitian matrices and compact operators in the spectral norm.

Theorem 6.1. *Let $C \in \mathcal{K}(\mathcal{H})$ such that C is a block-diagonal operator, that is*

$$C = \begin{pmatrix} C_1 & 0 & 0 & \cdots \\ 0 & C_2 & 0 & \cdots \\ 0 & 0 & C_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $C_i \in M^h_{n_i}(\mathbb{C})$, for each $i \in \mathbb{N}$. Then, there exists $D \in \mathcal{D}(\mathcal{K}(\mathcal{H}))$ such that $C + D$ is minimal.

Proof. For each $i \in \mathbb{N}$ there exists a minimizing $D_i \in \mathcal{D}(M^h_{n_i}(\mathbb{C}))$. That is

$$\|C_i + D_i\| \leq \|C_i + D_i'\| \ , \text{ for all } D_i' \in \mathcal{D}(M^h_{n_i}(\mathbb{C})).$$

We claim that the block-diagonal operator defined as

$$D = \begin{pmatrix} D_1 & 0 & 0 & \cdots \\ 0 & D_2 & 0 & \cdots \\ 0 & 0 & D_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is a minimizant for C. Indeed, it is trivial to observe that is diagonal since each block D_i is a diagonal matrix. It remains to prove compacity and minimality.

- Minimality: Let $D' \in \mathcal{D}(\mathcal{K}(\mathcal{H}))$. It can be written in the same block notation of C as

$$D' = \begin{pmatrix} D'_1 & 0 & 0 & \cdots \\ 0 & D'_2 & 0 & \cdots \\ 0 & 0 & D'_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \text{ with } D'_i \in \mathcal{D}(M^h_{n_i}(\mathbb{C})), n_i \in \mathbb{N}.$$

  Then,

$$\|C + D'\| = \sup_{i \in \mathbb{N}} \|C_i + D'_i\| \geq \sup_{i \in \mathbb{N}} \|C_i + D_i\| = \|C + D\|.$$

- Compacity: by minimality, $\|C_i + D_i\| \leq \|C_i\|$ for each $i \in \mathbb{N}$, then

$$\|D_i\| \leq \|C_i + D_i\| + \|C_i\| \leq 2\|C_i\| \to 0,$$

  when $i \to \infty$ (since C is compact). Therefore, $\lim\limits_{i \to \infty} D_i = 0$ and D is also compact.

□

Remark 6.2. In Theorem 6.1 the operator norm of $C + D$ is $\sup\{\|C_i + D_i\| : i \in \mathbb{N}\}$, which is clearly attained at any $i_0 \in \mathbb{N}$, since $\lim\limits_{i \to \infty} \|C_i + D_i\| = 0$.

Lemma 6.3. *(due to Prof. Varela) Any tridiagonal Hermitian matrix M of $n \times n$ with zero diagonal is minimal in the spectral norm.*

Proof. Let $M \in M^h_n(\mathbb{C})$ be a tridiagonal matrix defined as a polar way, that is,

$$M = \begin{pmatrix} 0 & a_1 e^{it_1} & 0 & \cdots & 0 \\ a_1 e^{-it_1} & 0 & a_2 e^{it_2} & \cdots & 0 \\ 0 & a_2 e^{it_2} & 0 & a_3 e^{it_3} & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{n-1} e^{-it_{n-1}} & 0 \end{pmatrix}, \tag{6.1}$$

with $a_i, t_i \in \mathbb{R}$ for all $1 \leq i \leq n$, and consider a unitary matrix U, given by

$$U = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & e^{it_1 + i\frac{\pi}{2}} & 0 & \cdots & 0 \\ 0 & 0 & e^{i(t_1+t_2)+i\pi} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e^{\left[i(t_1+\ldots+t_{n-1}) + \frac{n-1}{2} i\pi\right]} \end{pmatrix}.$$

Then,

$$UMU^* = \begin{pmatrix} 0 & ia_1 & 0 & \cdots & 0 \\ -ia_1 & 0 & ia_2 & \cdots & 0 \\ 0 & -ia_2 & 0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & ia_{n-1} \\ 0 & 0 & \cdots & -ia_{n-1} & 0 \end{pmatrix}$$

with $Re(UMU_{ij}) = 0$ for all $i, j \leq n$ and $Diag(M') = 0$. Thus, by Theorem 8 in [28], M' is minimal. Therefore,

$$\|M\| = \|U^*M'U\| = \|M'\| \leq \|M' + D\| = \|M + U^*DU\| = \|M + D\|,$$

where last equality is due to U is diagonal. Then,

$$\|M\| \leq \|M + D\| \text{ for all } D \text{ diagonal.}$$

□

Proposition 6.4 (Unicity). *Let M be a tridiagonal Hermitian matrix as (6.1) such that $a_i \neq 0$ for all $1 \leq i \leq n$. Then, 0 is the unique minimizing real diagonal for M.*

Proof. Let M as in the statement and $\lambda = \|M\|$. By proposition 6.3 M is minimal and $\pm\lambda \in \sigma(M)$. Moreover, simple calculations show that if $x = (x_1, x_2, ..., x_n)$ is an eigenvector associated to λ, then $y = (-x_1, x_2, ..., (-1)^n x_n)$ is an eigenvector associated to $-\lambda$ (and this holds for all $\mu \in \sigma(M)$, so every eigenspace of M has multiplicity one). In particular, $x_1 \neq 0$ implies $x \neq 0$ and $y_k = (-1)^k x_k \neq 0$ for all $1 < k \leq n$. Then $|y_k|^2 = |x_k|^2$ for each $1 \leq k \leq n$ and therefore,

$$x \circ \overline{x} = (|x_1|^2, |x_2|^2, ..., |x_n|^2) = y \circ \overline{y},$$

By [Theorem 10,[28]] M has only one minimizing real diagonal D, which is $D = 0$.

□

Finally, we use Lemma 6.3 to state the minimality of tridiagonal compact operators, since each matrix M of $n \times n$ can be seen as a finite rank operator in $\mathcal{K}(\mathcal{H})$, that is $M = P_{\{e_1,...,e_n\}} M P_{\{e_1,...,e_n\}}$, with $M_{ij} = \langle Me_i, e_j \rangle$ for each $i, j \in \mathbb{N}$ and $P_{\{e_1,...,e_n\}}$ is the orthogonal projection to the subspace span$\{e_1, ..., e_n\}$.

Proposition 6.5. *If $C \in \mathcal{K}(\mathcal{H})$ is a minimal Hermitian tridiagonal operator with $C_{i(i+1)} \neq 0$ for all $i \in \mathbb{N}$. Then, the following statements are equivalent:*

1. *C is a minimal operator in $\mathcal{K}(\mathcal{H})$.*
2. $\mathrm{Diag}(C) = 0$.

Proof. (1)⇒(2): Let $\{C_n\}_{n\in\mathbb{N}}$ be a Hermitian tridiagonal matrix sequence such that

$$\lim_{n\to\infty} C_n = C \text{ y } \mathrm{Diag}(C_n) = 0.$$

Each $C_n \in \mathcal{K}(\mathcal{H})$, since $rank(C_n) < \infty$, and by Proposition 6.3 all are minimal. Then

$$\begin{aligned} \|C\| &= \left\|\lim_{n\to\infty} C_n\right\| = \lim_{n\to\infty} \|C_n\| \\ &\leq \lim_{n\to\infty} \|C_n + D\| = \left\|\lim_{n\to\infty} C_n + D\right\| = \|C + D\| \end{aligned}$$

for each $D \in \mathcal{D}(\mathcal{K}(\mathcal{H}))$. Then, it is evident that $\mathrm{Diag}(C) = 0$.

(2)⇒(1): If C is a tridiagonal operator with zero diagonal, then P_nCP_n is a tridiagonal matrix with zero diagonal for all $n \in \mathbb{N}$ ($P_n = P_{\{e_1,\ldots,e_n\}}$). Then each P_nCP_n is a minimal matrix in $M_n(\mathbb{C})^h$. On the other hand, every $D \in \mathcal{D}(\mathcal{K}(\mathcal{H})^h)$ can be obtained as $\lim\limits_{n\to\infty} D_n$, with $D_n \in D(M_n(\mathbb{C})^h)$ for all $n \in \mathbb{N}$ (and $\|D_n\| \to 0$ when $n \to \infty$). Then, for all $D \in \mathcal{D}(\mathcal{K}(\mathcal{H})^h)$

$$\|C\| = \lim_{n\to\infty} \|P_nCP_n\| \leq \lim_{n\to\infty} \|P_nCP_n + D_n\| = \|C + D\|.$$

Therefore, C is minimal.

□

Observe that implication (1)⇒(2) holds without the requirement $C_{i(i+1)} \neq 0$.

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