On A-parallelism and A-Birkhoff-James orthogonality of operators

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Abstract. In this paper, we establish several characterizations of the A-parallelism of bounded linear operators with respect to the seminorm induced by a positive operator A acting on a complex Hilbert space. Among other things, we investigate the relationship between A-seminorm-parallelism and A-Birkhoff-James orthogonality of A-bounded operators. In particular, we characterize A-bounded operators which satisfy the A-Daugavet equation. In addition, we relate the A-Birkhoff-James orthogonality of operators and distance formulas and we give an explicit formula of the center mass for A-bounded operators. Some other related results are also discussed.

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1. Introduction and Preliminaries

Let $\mathbb{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators acting on a non trivial complex Hilbert space $\mathcal{H}$ with an inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$. The symbol $I_{\mathcal{H}}$ stands for the identity operator on $\mathcal{H}$ (or I if no confusion arises). In all that follows, by an operator we mean a bounded linear operator. The range of every operator is denoted by $\mathcal{R}(T)$, its null space by $\mathcal{N}(T)$ and T^* is the adjoint of T. If $T, S \in \mathbb{B}(\mathcal{H})$, we write $T \geq S$ whenever $\langle Tx, x\rangle \geq \langle Sx, x\rangle$ for all $x \in \mathcal{H}$. An element $A \in \mathbb{B}(\mathcal{H})$ such that $A \geq 0$ is called positive. For every $A \geq 0$, there exists a unique positive $A^{1/2} \in \mathbb{B}(\mathcal{H})$ such that $A = (A^{1/2})^2$. For the rest of this article, we assume that $A \in \mathbb{B}(\mathcal{H})$ is a positive nonzero operator, which clearly induces the following semi-inner product

$$\langle \cdot, \cdot \rangle_A : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{C},\ (x, y) \longmapsto \langle x, y\rangle_A := \langle Ax, y\rangle.$$

Notice that the induced seminorm is given by $\|x\|_A = \sqrt{\langle x, x\rangle_A}$, for every $x \in \mathcal{H}$. This makes $\mathcal{H}$ into a semi-Hilbert space. One can check that $\| \cdot \|_A$ is a norm on $\mathcal{H}$ if and only if A is injective, and that $(\mathcal{H}, \| \cdot \|_A)$ is complete if and only if $\mathcal{R}(A)$ is closed. The semi-inner product $\langle\cdot,\cdot\rangle_A$ induces an inner product on the quotient space $\mathcal{H}/\mathcal{N}(A)$ defined as

$$[\overline{x},\overline{y}] = \langle Ax, y\rangle, \quad \forall\, \overline{x},\overline{y} \in \mathcal{H}/\mathcal{N}(A).$$

Notice that $(\mathcal{H}/\mathcal{N}(A), [\cdot,\cdot])$ is not complete unless $\mathcal{R}(A)$ is a closed subset of $\mathcal{H}$. However, a canonical construction due to L. de Branges and J. Rovnyak in [9] (see also [14]) shows that the completion of $\mathcal{H}/\mathcal{N}(A)$ under the inner product $[\cdot,\cdot]$ is isometrically isomorphic to the Hilbert space $\mathcal{R}(A^{1/2})$ with the inner product

$$\langle A^{1/2}x, A^{1/2}y\rangle_{\mathbf{R}(A^{1/2})} := \langle P_{\overline{\mathcal{R}(A)}}x, P_{\overline{\mathcal{R}(A)}}y\rangle,\ \forall\, x,y \in \mathcal{H}, \tag{1.1}$$

where $P_{\overline{\mathcal{R}(A)}}$ denotes the orthogonal projection onto $\overline{\mathcal{R}(A)}$. For the sequel, the Hilbert space $\big(\mathcal{R}(A^{1/2}), \langle\cdot,\cdot\rangle_{\mathbf{R}(A^{1/2})}\big)$ will be denoted by $\mathbf{R}(A^{1/2})$. By using (1.1), one can check that

$$\langle Ax, Ay\rangle_{\mathbf{R}(A^{1/2})} = \langle x, y\rangle_A, \quad \forall\, x,y \in \mathcal{H},$$

which, in turn, implies that

$$\|Ax\|_{\mathbf{R}(A^{1/2})} = \|x\|_A, \quad \forall\, x \in \mathcal{H}. \tag{1.2}$$

We refer the reader to [4] and the references therein for more information concerning the Hilbert space $\mathbf{R}(A^{1/2})$.

For $T \in \mathbb{B}(\mathcal{H})$, an operator $S \in \mathbb{B}(\mathcal{H})$ is said an A-adjoint operator of T if the identity $\langle Tx, y\rangle_A = \langle x, Sy\rangle_A$ holds for every $x, y \in \mathcal{H}$, or equivalently, S is solution of the operator equation $AX = T^*A$. Notice that this kind of equation can be investigated by using the following well-known theorem due to Douglas (for its proof see [12]).

Theorem A. *If $T, S \in \mathbb{B}(\mathcal{H})$, then the following statements are equivalent:*

(i) $\mathcal{R}(S) \subseteq \mathcal{R}(T)$.
(ii) $TD = S$ *for some* $D \in \mathbb{B}(\mathcal{H})$.
(iii) *There exists* $\lambda > 0$ *such that* $\|S^*x\| \leq \lambda\|T^*x\|$ *for all* $x \in \mathcal{H}$.

If one of these conditions holds, then there exists a unique solution of the operator equation $TX = S$, denoted by Q, such that $\mathcal{R}(Q) \subseteq \overline{\mathcal{R}(T^*)}$. Such Q is called the reduced solution of $TX = S$.

If we denote by $\mathbb{B}_A(\mathcal{H})$ and $\mathbb{B}_{A^{1/2}}(\mathcal{H})$ the sets of all operators that admit A-adjoints and $A^{1/2}$-adjoints, respectively, then an application of Theorem A gives

$$\mathbb{B}_A(\mathcal{H}) = \big\{T \in \mathbb{B}(\mathcal{H});\ \mathcal{R}(T^*A) \subseteq \mathcal{R}(A)\big\},$$

and

$$\mathbb{B}_{A^{1/2}}(\mathcal{H}) = \big\{T \in \mathbb{B}(\mathcal{H});\ \exists\, c > 0;\ \|Tx\|_A \leq c\|x\|_A,\ \forall x \in \mathcal{H}\big\}.$$

Operators in $\mathbb{B}_{A^{1/2}}(\mathcal{H})$ are called A-bounded. Notice that $\mathbb{B}_A(\mathcal{H})$ and $\mathbb{B}_{A^{1/2}}(\mathcal{H})$ are two subalgebras of $\mathbb{B}(\mathcal{H})$ which are, in general, neither closed nor dense in $\mathbb{B}(\mathcal{H})$ (see [2]). Moreover, the following inclusions $\mathbb{B}_A(\mathcal{H}) \subseteq \mathbb{B}_{A^{1/2}}(\mathcal{H}) \subseteq \mathbb{B}(\mathcal{H})$ hold and are in general proper (see [15]).

Let $T \in \mathbb{B}_A(\mathcal{H})$. The reduced solution of the equation $AX = T^*A$ will be denoted by $T^{\sharp_A}$. Note that, $T^{\sharp_A} = A^\dagger T^* A$. Here $A^\dagger$ is the Moore-Penrose inverse of A. We mention that if $T \in \mathbb{B}_A(\mathcal{H})$, then $T^{\sharp_A} \in \mathbb{B}_A(\mathcal{H})$ and $(T^{\sharp_A})^{\sharp_A} = P_{\overline{\mathcal{R}(A)}} T P_{\overline{\mathcal{R}(A)}}$. For more results concerning $T^{\sharp_A}$ see [2, 3]. It is useful to recall that an operator $T \in \mathbb{B}_A(\mathcal{H})$ is called A-normal if $TT^{\sharp_A} = T^{\sharp_A}T$ (see [4, 8]). Notice that T is A-normal if and only if $\mathcal{R}(TT^{\sharp_A}) \subseteq \overline{\mathcal{R}(A)}$ and $\|T^{\sharp_A}x\|_A = \|Tx\|_A$ for all $x \in \mathcal{H}$ (see [23]). Now, it is well-known that $\langle \cdot, \cdot \rangle_A$ induces on $\mathbb{B}_{A^{1/2}}(\mathcal{H})$ the following seminorm:

$$\|T\|_A := \sup_{\substack{x \in \overline{\mathcal{R}(A)},\\ x \neq 0}} \frac{\|Tx\|_A}{\|x\|_A} = \sup\left\{\|Tx\|_A\,;\ x \in \mathcal{H},\ \|x\|_A = 1\right\} < \infty. \tag{1.3}$$

It can be observed that for $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$, $\|T\|_A = 0$ if and only if $AT = 0$. Notice that it was proved in [13] that for $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$ we have

$$\|T\|_A = \sup\left\{|\langle Tx, y\rangle_A|\,;\ x, y \in \mathcal{H},\ \|x\|_A = \|y\|_A = 1\right\}. \tag{1.4}$$

It can be verified that, for $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$, we have $\|Tx\|_A \leq \|T\|_A \|x\|_A$ for all $x \in \mathcal{H}$. This implies that, for $T, S \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$, we have $\|TS\|_A \leq \|T\|_A \|S\|_A$. In addition, we have $\|T^{\sharp_A}T\|_A = \|TT^{\sharp_A}\|_A = \|T\|_A^2 = \|T^{\sharp_A}\|_A^2$ for all $T \in \mathbb{B}_A(\mathcal{H})$ (see [3, Proposition 2.3.]). Notice that it may happen that $\|T\|_A = +\infty$ for some $T \in \mathbb{B}(\mathcal{H})$ (see [15]). For more details concerning A-bounded operators, see [4] and the references therein. Recently, A. Saddi generalized in [23] the concept of the numerical radius of Hilbert space operators and defined the A-numerical radius of an operator $T \in \mathbb{B}(\mathcal{H})$ by

$$\omega_A(T) = \sup\left\{|\langle Tx, x\rangle_A|\ ;\ x \in \mathcal{H}, \|x\|_A = 1\right\} = \sup\left\{|\lambda|\ ;\ \lambda \in W_A(T)\right\},$$

where $W_A(T)$ denotes the A-numerical range of T which was firstly defined by Baklouti et al. in [7] as

$$W_A(T) = \left\{\langle Tx, x\rangle_A\ ;\ x \in \mathcal{H}, \|x\|_A = 1\right\}.$$

If $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$ then $\omega_A(T) < +\infty$ and

$$\frac{1}{2}\|T\|_A \leq \omega_A(T) \leq \|T\|_A.$$

Very recently, the A-Davis-Wielandt radius of an operator $T \in \mathbb{B}(\mathcal{H})$ is defined, as in [18], by

$$d\omega_A(T) = \sup\left\{\sqrt{|\langle Tx, x\rangle_A|^2 + \|Tx\|_A^4}\,;\ x \in \mathcal{H},\ \|x\|_A = 1\right\}.$$

Notice that it was shown in [18], that for $T \in \mathbb{B}(\mathcal{H})$, $d\omega_A(T)$ can be equal to $+\infty$. However, if $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$, then we have

$$\max\{\omega_A(T), \|T\|_A^2\} \leq d\omega_A(T) \leq \sqrt{\omega_A(T)^2 + \|T\|_A^4} < \infty.$$

Recently, the concept of the A-spectral radius of A-bounded operators has been introduced by the third author in [15] as follows:

$$r_A(T) := \inf_{n \geq 1} \|T^n\|_A^{\frac{1}{n}} = \lim_{n \to +\infty} \|T^n\|_A^{\frac{1}{n}}. \tag{1.5}$$

We note here that the second equality in (1.5) is also proved in [15, Theorem 1]. An operator $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$ is said to be A-normaloid if $r_A(T) = \|T\|_A$. Moreover, T is called A-spectraloid if $r_A(T) = \omega_A(T)$. It was shown in [15] that for every A-normaloid operator $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$ we have

$$r_A(T) = \omega_A(T) = \|T\|_A. \tag{1.6}$$

Obviously, (1.6) implies that every A-normaloid operator is A-spectraloid.

Throughout this paper, let $\mathbb{T}$ denote the unit cycle of the complex plane, i.e. $\mathbb{T} = \{\lambda \in \mathbb{C};\ |\lambda| = 1\}$. Recall from [18] that an operator $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$ is said to be A-seminorm-parallel to an operator $S \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$, in short $T \parallel_A S$, if there exists some $\lambda \in \mathbb{T}$ such that $\|T + \lambda S\|_A = \|T\|_A + \|S\|_A$. If $A = I$, then $\parallel_I$ will simply denoted by $\parallel$. Recall also from [27] that an element $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$ is said to be A-Birkhoff-James orthogonal to another element $S \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$, denoted by $T \perp_A^{BJ} S$, if

$$\|T + \gamma S\|_A \geq \|T\|_A, \quad \text{for all } \gamma \in \mathbb{C}.$$

Very recently, the A-Birkhoff-James orthogonality of A-bounded operators has been investigated by Sen et al. in [24]. We mention also here that several results covering some classes of Hilbert space operators were extended to A-bounded operators (see, e.g., [10, 14, 15, 16, 18, 21, 27] and the references therein).

The following lemma will be used in due course of time. Notice that the proof of the assertion (i) can be found in [4]. Further, for the proof of the assertions (ii) and (iii), we refer to [15]. In addition, the assertion (iv) has been proved in [21]. Finally, the proof of last assertion can be found in [18].

Lemma B. *Let $T \in \mathbb{B}(\mathcal{H})$. Then $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$ if and only if there exists a unique $\widetilde{T} \in \mathbb{B}(\mathbf{R}(A^{1/2}))$ such that $Z_A T = \widetilde{T} Z_A$. Here, $Z_A : \mathcal{H} \to \mathbf{R}(A^{1/2})$ is defined by $Z_A x = Ax$. Moreover, the following properties hold*

(i) $\|T\|_A = \|\widetilde{T}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))}$.
(ii) $r_A(T) = r(\widetilde{T})$.
(iii) $\overline{W_A(T)} = \overline{W(\widetilde{T})}$.
(iv) *If $T \in \mathbb{B}_A(\mathcal{H})$, then $\widetilde{T^{\sharp_A}} = (\widetilde{T})^*$.*
(v) *If $T, S \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$, then $T \parallel_A S$ if and only if $\widetilde{T} \parallel \widetilde{S}$.*

The remainder of the paper is organized as follows. In Section 2, we present different characterizations of the notion of A-seminorm-parallelism. Some of the obtained results cover and extend the work of Zamani et al. [26]. In particular, we investigate when the A-Davis-Wielandt radius of an operator coincides with its upper bound. In section 3, we give another characterizations of A-seminorm-parallelism related to A-Birkhoff-James orthogonality. Finally, section 4 is devoted to obtain some formulas for the A-center of mass of A-bounded operators using well-known distance formulas.

2. A-seminorm-parallelism

Our starting point in the present section is the following examples of seminorm-parallelism in semi-Hilbert spaces.

Examples 2.1. (1) *Let $T, S \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$ be linearly dependent operators. Then $T \parallel_A S$ (see* [18, Example 3]*).*

(2) *Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ be operators acting on $\mathbb{C}^2$. Then for $\lambda = 1$, simple computations show that*

$$\|T + \lambda I\|_A = \|T\|_A + \|I\|_A = 2.$$

Hence $T \parallel_A I$.

(3) *Let $\lambda > 0$ and $A, T, S : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be such that*

$$S(\overline{x}) = (\lambda x_1, \lambda x_2, x_3, x_4, \dots), \quad T(\overline{x}) = (0, \lambda x_2, x_3, x_4, \dots)$$

and

$$A(\overline{x}) = (0, x_2, 0, 0, \dots),$$

for every $\overline{x} = (x_1, x_2, \dots, x_n, \dots) \in \ell^2(\mathbb{N})$, where $\mathbb{N}$ denotes the set of all positive integers. Clearly, $A \geq 0$. Further, it can be observed that $\|T\|_A = \|S\|_A = \lambda$. Now, let $\{e_j\}_{j\in\mathbb{N}}$ be the canonical orthogonal basis of $\mathcal{H} = \ell^2(\mathbb{N})$. Then, we have

$$\|(T + S)(e_2)\|_A^2 = 4\lambda^2.$$

Thus, $2\lambda \leq \|T + S\|_A \leq \|T\|_A + \|S\|_A = 2\lambda$. Therefore $T \parallel_A S$.

In the following proposition, we state some basic properties of operator seminorm-parallelism in $\mathbb{B}_{A^{1/2}}(\mathcal{H})$.

Proposition 2.1. *Let $T, S \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$. The following statements are equivalent:*

(1) *$T \parallel_A S$.*
(2) *$\alpha T \parallel_A \alpha S$ for every $\alpha \in \mathbb{C} \setminus \{0\}$.*
(3) *$\beta T \parallel_A \gamma S$ for every $\beta, \gamma \in \mathbb{R} \setminus \{0\}$*

Proof. Notice that equivalence (1)⇔(2) follows immediately from the definition of A-operator parallelism.
(1)⇒(3) Assume that $T \parallel_A S$. Thus $\|T + \lambda S\|_A = \|T\|_A + \|S\|_A$ for some $\lambda \in \mathbb{T}$. Let $\beta, \gamma \in \mathbb{R} \setminus \{0\}$. We suppose that $\beta \geq \gamma > 0$. Hence, we see that

$$\begin{aligned} \|\beta T\|_A + \|\gamma S\|_A &\geq \|\beta T + \lambda(\gamma S)\|_A \\ &= \|\beta(T + \lambda S) - (\beta - \gamma)(\lambda S)\|_A \\ &\geq \|\beta(T + \lambda S)\|_A - \|(\beta - \gamma)\lambda S\|_A \\ &= \beta\|T + \lambda S\|_A - (\beta - \gamma)\|S\|_A \\ &= \beta(\|T\|_A + \|S\|_A) - (\beta - \gamma)\|S\|_A \\ &= \|\beta T\|_A + \|\gamma S\|_A. \end{aligned}$$

So, $\|\beta T + \lambda(\gamma S)\|_A = \|\beta T\|_A + \|\gamma S\|_A$ for some $\lambda \in \mathbb{T}$. Thus, $\beta T \parallel_A \gamma S$.
(3)⇒(1) is trivial. □

The following lemma is useful in the sequel.

Lemma 2.1. *Let $T, S \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$. Then the following statements are equivalent:*

(i) $T \parallel_A S$.
(ii) *There exist a sequence of A-unit vectors $\{x_n\}$ in $\mathcal{H}$ (i.e. $\|x_n\|_A = 1$ for all n) and $\lambda \in \mathbb{T}$ such that*

$$\lim_{n\to+\infty} \langle Tx_n, Sx_n\rangle_A = \lambda \|T\|_A \, \|S\|_A.$$

In order to prove Lemma 2.1, we need the following result.

Theorem C. *([18]) Let $T, S \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$. Then, $T \parallel_A S$ if and only if there exists a sequence of A-unit vectors $\{x_n\}$ in $\mathcal{H}$ such that*

$$\lim_{n\to+\infty} |\langle Tx_n, Sx_n\rangle_A| = \|T\|_A \|S\|_A. \tag{2.1}$$

Remark 2.1. *In addition, if $\|T\|_A\|S\|_A \neq 0$ and $\{x_n\}$ is a sequence of A-unit vectors in $\mathcal{H}$ satisfying (2.1), then it also satisfies*

$$\lim_{n\to+\infty} \|Tx_n\|_A = \|T\|_A \qquad \text{and} \qquad \lim_{n\to+\infty} \|Sx_n\|_A = \|S\|_A.$$

Indeed, for any $\varepsilon > 0$ and n large enough we have

$$\|T\|_A\|S\|_A \geq \|S\|_A\|Tx_n\|_A \geq |\langle Tx_n, Sx_n\rangle_A| \geq \|S\|_A\|T\|_A - \varepsilon.$$

Hence, $\lim_{n\to+\infty} \|Tx_n\|_A = \|T\|_A$. Analogously, by changing the roles between T and S, we obtain $\lim_{n\to+\infty} \|Sx_n\|_A = \|S\|_A$.

Now, we state the proof of Lemma 2.1.

Proof of Lemma 2.1. Assume that $T \parallel_A S$, then by Theorem C there exists a sequence of A-unit vectors $\{x_n\}$ in $\mathcal{H}$ such that

$$\lim_{n\to+\infty} |\langle Tx_n, Sx_n\rangle_A| = \|T\|_A \, \|S\|_A. \tag{2.2}$$

Suppose that $\|T\|_A \, \|S\|_A \neq 0$ (otherwise the desired assertion holds trivially). Since $\mathbb{T}$ is a compact subset of $\mathbb{C}$, then by taking a further subsequence we may assume that there is some $\lambda \in \mathbb{T}$ such that

$$\lim_{n\to+\infty} \frac{\langle Tx_n, Sx_n\rangle_A}{|\langle Tx_n, Sx_n\rangle_A|} = \lambda.$$

So, by using (2.2) we get

$$\lim_{n\to+\infty} \langle Tx_n, Sx_n\rangle_A = \lim_{n\to+\infty} \frac{\langle Tx_n, Sx_n\rangle_A}{|\langle Tx_n, Sx_n\rangle_A|} |\langle Tx_n, Sx_n\rangle_A| = \lambda \|T\|_A\|S\|_A.$$

The converse implication follows immediately by applying Theorem C. $\square$

Our next goal is to characterize the A-seminorm-parallelism of operators in $\mathbb{B}_A(\mathcal{H})$. To achieve this goal, we shall need some lemmas. In what follows $\sigma(T)$, $\sigma_a(T)$, $r(T)$ and $W(T)$ stand for the spectrum, the approximate spectrum, the spectral radius and the numerical range of an arbitrary element $T \in \mathbb{B}(\mathcal{H})$, respectively.

Lemma 2.2. *([20, Theorem 1.2-1]) Let $T \in \mathbb{B}(\mathcal{H})$. Then, $\sigma(T) \subseteq \overline{W(T)}$.*

Lemma 2.3. *([22, Theorem 3.3.6]) Let $T \in \mathbb{B}(\mathcal{H})$ be a normal operator. Then there exists a state ψ (i.e. a functional $\psi : \mathbb{B}(\mathcal{H}) \to \mathbb{C}$ with $\|\psi\| = 1$ and $\psi(T^*T) \geq 0$ for all $T \in \mathbb{B}(\mathcal{H})$) such that $\psi(T) = \|T\|$.*

Now, we are in a position to prove the following result.

Theorem 2.1. *Let $T, S \in \mathbb{B}_A(\mathcal{H})$. Then the following statements are equivalent:*

(1) $T \parallel_A S$.
(2) $r_A(S^{\sharp_A}T) = \|S^{\sharp_A}T\|_A = \|T^{\sharp_A}S\|_A = \|T\|_A\,\|S\|_A$.
(3) $T^{\sharp_A}T \parallel_A T^{\sharp_A}S$ *and* $\|T^{\sharp_A}S\|_A = \|T\|_A\,\|S\|_A$.
(4) $\|T^{\sharp_A}(T+\lambda S)\|_A = \|T\|_A(\|T\|_A + \|S\|_A)$ *for some* $\lambda \in \mathbb{T}$.

Proof. (1)⇒(2) Assume that $T \parallel_A S$. If $AT = 0$ or $AS = 0$, then by using (1.4) we can verify that the assertion (2) holds. Suppose that $AT \neq 0$ and $AS \neq 0$, i.e. $\|T\|_A \neq 0$ and $\|S\|_A \neq 0$. Since $T \parallel_A S$, then by Lemma 2.1, there exists a sequence of A-unit vectors $\{x_n\}$ in $\mathcal{H}$ satisfying

$$\lim_{n\to+\infty} \langle Tx_n, Sx_n\rangle_A = \lambda \|T\|_A\,\|S\|_A, \tag{2.3}$$

for some $\lambda \in \mathbb{T}$. This implies that

$$\lim_{n\to+\infty} \Re\left(\langle Tx_n, \lambda Sx_n\rangle_A\right) = \|T\|_A\,\|S\|_A, \tag{2.4}$$

where $\Re(z)$ denotes the real part of $z \in \mathbb{C}$. Moreover, by using the Cauchy-Schwarz inequality it follows from

$$\|T\|_A\,\|S\|_A = \lim_{n\to+\infty} |\langle Tx_n, Sx_n\rangle_A| \leq \lim_{n\to+\infty} \|Tx_n\|_A\,\|S\|_A \leq \|T\|_A\,\|S\|_A.$$

This immediately implies that $\lim\limits_{n\to+\infty} \|Tx_n\|_A = \|T\|_A$. In addition, by similar arguments as above, we obtain $\lim\limits_{n\to+\infty} \|Sx_n\|_A = \|S\|_A$. So, by taking into consideration (2.4), we see that

$$\begin{aligned}
\|T\|_A + \|S\|_A &\geq \|T+\lambda S\|_A \\
&\geq \left(\lim_{n\to+\infty} \|(T+\lambda S)x_n\|_A^2\right)^{\frac{1}{2}} \\
&\geq \left(\lim_{n\to+\infty} \left[\|Tx_n\|_A^2 + 2\Re\left(\langle Tx_n, \lambda Sx_n\rangle_A\right) + \|Sx_n\|_A^2\right]\right)^{\frac{1}{2}} \\
&= \left(\|T\|_A^2 + 2\|S\|_A\|T\|_A + \|S\|_A^2\right)^{\frac{1}{2}} = \|T\|_A + \|S\|_A.
\end{aligned}$$

Thus, we infer that $\|T+\lambda S\|_A = \|T\|_A + \|S\|_A$. Hence, it can be observed that

$$\begin{aligned}(\|T\|_A + \|S\|_A)^2 &= \|T+\lambda S\|_A^2 \\ &= \|(T+\lambda S)^{\sharp_A}(T+\lambda S)\|_A \\ &\leq \|T^{\sharp_A}T\|_A + \|\lambda T^{\sharp_A}S\|_A + \|\overline{\lambda}S^{\sharp_A}T\|_A + \|S^{\sharp_A}S\|_A \\ &\leq \|T\|_A^2 + 2\|T\|_A\,\|S\|_A + \|S\|_A^2 \\ &= (\|T\|_A + \|S\|_A)^2.\end{aligned}$$

This implies that $\|T^{\sharp_A}S\|_A + \|S^{\sharp_A}T\|_A = 2\|T\|\,\|S\|$. On the other hand, one observes that $P_{\overline{\mathcal{R}(A)}}A = AP_{\overline{\mathcal{R}(A)}} = A$. Moreover, by (1.4), we see that

$$\begin{aligned}\|T^{\sharp_A}S\|_A &= \|S^{\sharp_A}P_{\overline{\mathcal{R}(A)}}TP_{\overline{\mathcal{R}(A)}}\|_A \\ &= \sup\left\{|\langle AP_{\overline{\mathcal{R}(A)}}x, (S^{\sharp_A}P_{\overline{\mathcal{R}(A)}}T)^{\sharp_A}y\rangle|\,;\; x,y\in\mathcal{H},\; \|x\|_A = \|y\|_A = 1\right\} \\ &= \sup\left\{|\langle S^{\sharp_A}P_{\overline{\mathcal{R}(A)}}Tx, y\rangle_A|\,;\; x,y\in\mathcal{H},\; \|x\|_A = \|y\|_A = 1\right\} \\ &= \sup\left\{|\langle AP_{\overline{\mathcal{R}(A)}}Tx, Sy\rangle|\,;\; x,y\in\mathcal{H},\; \|x\|_A = \|y\|_A = 1\right\} \\ &= \sup\left\{|\langle S^{\sharp_A}Tx, y\rangle_A|\,;\; x,y\in\mathcal{H},\; \|x\|_A = \|y\|_A = 1\right\} \\ &= \|S^{\sharp_A}T\|_A.\end{aligned}$$

Hence, we deduce that

$$\|S^{\sharp_A}T\|_A = \|T^{\sharp_A}S\|_A = \|T\|_A\,\|S\|_A. \tag{2.5}$$

Moreover, by using the Cauchy-Shwarz inequality, we see that

$$\begin{aligned}\|T\|_A\,\|S\|_A &= \lim_{n\to+\infty}|\langle Tx_n, Sx_n\rangle_A| \\ &\leq \lim_{n\to+\infty}\|S^{\sharp_A}Tx_n\|_A \\ &\leq \|S^{\sharp_A}T\|_A = \|T\|_A\,\|S\|_A,\end{aligned}$$

where the last equality follows from (2.5). So, we have

$$\lim_{n\to+\infty}\|S^{\sharp_A}Tx_n\|_A = \|T\|_A\,\|S\|_A. \tag{2.6}$$

On the other hand, it can be observed that

$$\begin{aligned}\|(S^{\sharp_A}T - \lambda\|T\|_A\,\|S\|_A I)x_n\|_A^2 &= \|S^{\sharp_A}Tx_n\|_A^2 + \|T\|_A^2\,\|S\|_A^2 \\ &\quad - 2\|T\|_A\,\|S\|_A\Re\left(\overline{\lambda}\langle Tx_n, Sx_n\rangle_A\right).\end{aligned}$$

So, by using (2.3) together with (2.6) we get

$$\lim_{n\to+\infty}\left\|\left(S^{\sharp_A}T - \lambda\|T\|_A\,\|S\|_A I\right)x_n\right\|_A = 0.$$

This implies, thought (1.2), that

$$\lim_{n\to+\infty}\left\|A\left(S^{\sharp_A}T - \lambda\|T\|_A\,\|S\|_A I\right)x_n\right\|_{\mathbf{R}(A^{1/2})} = 0,$$

So, by using Lemma B we get

$$\lim_{n\to+\infty}\left\|\left((\widetilde{S})^*\widetilde{T}-\lambda\|T\|_A\,\|S\|_A I_{\mathbf{R}(A^{1/2})}\right)Ax_n\right\|_{\mathbf{R}(A^{1/2})}=0.$$

Since $\|Ax_n\|_{\mathbf{R}(A^{1/2})}=\|x_n\|_A=1$. Then, $\lambda\|T\|_A\,\|S\|_A\in\sigma_a\left((\widetilde{S})^*\widetilde{T}\right)$. So,

$$\|T\|_A\,\|S\|_A\leq r\left((\widetilde{S})^*\widetilde{T}\right)=r(\widetilde{S^{\sharp_A}T})=r_A(S^{\sharp_A}T),$$

where the last equality follows from Lemma B. Further, clearly $r_A(S^{\sharp_A}T)\leq\|T\|_A\,\|S\|_A$. This proves, through (2.5), that

$$r_A(S^{\sharp_A}T)=\|T\|_A\,\|S\|_A=\|S^{\sharp_A}T\|_A=\|T^{\sharp_A}S\|_A,$$

as required.

(2) $\Rightarrow$ (1) Assume that (2) holds. Then, by applying Lemma B we can see that

$$r\left((\widetilde{S})^*\widetilde{T}\right)=\|T\|_A\,\|S\|_A.$$

Hence, there exists $\lambda_0\in\sigma\left((\widetilde{S})^*\widetilde{T}\right)$ such that $|\lambda_0|=\|T\|_A\,\|S\|_A$. So, by Lemma 2.2 together with Lemma B, we have

$$\lambda_0\in\overline{W\left((\widetilde{S})^*\widetilde{T}\right)}=\overline{W_A(S^{\sharp_A}T)}.$$

Thus there exists a sequence of A-unit vectors $\{x_n\}$ in $\mathcal{H}$ satisfying

$$\lim_{n\to+\infty}\langle Tx_n,Sx_n\rangle_A=\lambda_0.$$

This immediately proves the desired result by applying Theorem C.

(1)$\Rightarrow$(3) Assume that $T\parallel_A S$. Then, by Lemma 2.1 there exist a sequence of A-unit vectors $\{x_n\}$ in $\mathcal{H}$ and $\lambda\in\mathbb{T}$ such that

$$\lim_{n\to+\infty}\langle Tx_n,Sx_n\rangle_A=\lambda\|T\|_A\,\|S\|_A.$$

So by proceeding as in the implication (1)$\Rightarrow$(2), we obtain $\|T+\lambda S\|_A=\|T\|_A+\|S\|_A$ and $\|T^{\sharp_A}S\|_A=\|T\|_A\,\|S\|_A$. This implies, by Lemma B, that

$$\|\widetilde{T}+\lambda\widetilde{S}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))}=\|\widetilde{T}\|_{\mathbf{R}(A^{1/2})}+\|\widetilde{S}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))}\tag{2.7}$$

and

$$\|(\widetilde{T})^*\widetilde{S}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))}=\|\widetilde{T}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))}\,\|\widetilde{S}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))}.$$

Since $(\widetilde{T}+\lambda\widetilde{S})^*(\widetilde{T}+\lambda\widetilde{S})$ is a normal operator on the Hilbert space $\mathbf{R}(A^{1/2})$, then by using Lemma 2.3, we deduce that there exists a state ψ such that

$$\begin{aligned}\psi\left((\widetilde{T}+\lambda\widetilde{S})^*(\widetilde{T}+\lambda\widetilde{S})\right)&=\|(\widetilde{T}+\lambda\widetilde{S})^*(\widetilde{T}+\lambda\widetilde{S})\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))}\\&=\|\widetilde{T}+\lambda\widetilde{S}\|^2_{\mathbb{B}(\mathbf{R}(A^{1/2}))}\\&=\left(\|\widetilde{T}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))}+\|\widetilde{S}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))}\right)^2,\end{aligned}$$

where the last equality follows from (2.7). Thus

$$\begin{aligned}
&\left(\|\widetilde{T}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))} + \|\widetilde{S}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))}\right)^2 \\
&= \psi\left((\widetilde{T})^*\widetilde{T} + \lambda(\widetilde{T})^*\widetilde{S} + \overline{\lambda}(\widetilde{S})^*\widetilde{T} + (\widetilde{S})^*\widetilde{S}\right) \\
&\le \|(\widetilde{T})^*\widetilde{T}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))} + \|\lambda(\widetilde{T})^*\widetilde{S} + \overline{\lambda}(\widetilde{S})^*\widetilde{T}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))} + \|(\widetilde{S})^*\widetilde{S}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))} \\
&\le \|\widetilde{T}\|^2_{\mathbb{B}(\mathbf{R}(A^{1/2}))} + 2\|\widetilde{T}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))}\,\|\widetilde{S}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))} + \|\widetilde{S}\|^2_{\mathbb{B}(\mathbf{R}(A^{1/2}))} \\
&= \left(\|\widetilde{T}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))} + \|\widetilde{S}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))}\right)^2.
\end{aligned}$$

So $\psi\left((\widetilde{T})^*\widetilde{T}\right) = \|(\widetilde{T})^*\widetilde{T}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))}$ and $\psi\left(\lambda(\widetilde{T})^*\widetilde{S}\right) = \|(\widetilde{T})^*\widetilde{S}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))}$. Therefore, we have

$$\begin{aligned}
&\|(\widetilde{T})^*\widetilde{T}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))} + \|(\widetilde{T})^*\widetilde{S}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))} \\
&= \psi\left((\widetilde{T})^*\widetilde{T} + \lambda(\widetilde{T})^*\widetilde{S}\right) \\
&\le \|(\widetilde{T})^*\widetilde{T} + \lambda(\widetilde{T})^*\widetilde{S}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))} \\
&\le \|(\widetilde{T})^*\widetilde{T}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))} + \|(\widetilde{T})^*\widetilde{S}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))}.
\end{aligned}$$

Hence, we deduce that

$$\|(\widetilde{T})^*\widetilde{T} + \lambda(\widetilde{T})^*\widetilde{S}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))} = \|(\widetilde{T})^*\widetilde{T}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))} + \|(\widetilde{T})^*\widetilde{S}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))},$$

for some $\lambda \in \mathbb{T}$. Thus $(\widetilde{T})^*\widetilde{T} \parallel (\widetilde{T})^*\widetilde{S}$ which implies that $\widetilde{T^{\sharp_A}T} \parallel \widetilde{T^{\sharp_A}S}$. So, by Lemma B(v), $T^{\sharp_A}T \parallel_A T^{\sharp_A}S$.
(3)⇒(4) Follows obviously.
(4)⇒(1) Assume that $\|T^{\sharp_A}(T+\lambda S)\|_A = \|T\|_A(\|T\|_A+\|S\|_A)$ for some $\lambda \in \mathbb{T}$. Then we see that

$$\begin{aligned}
\|T\|_A(\|T\|_A + \|S\|_A) &\ge \|T^{\sharp_A}\|_A\|T + \lambda S\|_A \\
&\ge \|T^{\sharp_A}(T + \lambda S)\|_A \\
&= \|T\|_A(\|T\|_A + \|S\|_A).
\end{aligned}$$

So, if $AT \neq 0$, then $\|T + \lambda S\|_A = \|T\|_A + \|S\|_A$ which yields that $T \parallel_A S$. Furthermore, if $AT = 0$, then by taking (1.4) into account, we prove that $T \parallel_A S$. □

Corollary 2.1. *Let $T, S \in \mathbb{B}_A(\mathcal{H})$. The following conditions are equivalent:*

(1) $T \parallel_A S$.
(2) $\omega_A(S^{\sharp_A}T) = \|S^{\sharp_A}T\|_A = \|T^{\sharp_A}S\|_A = \|T\|_A\,\|S\|_A$.

To prove Corollary 2.1, we need the following Lemma.

Lemma D. *Let $T \in \mathbb{B}_A(\mathcal{H})$. Then T is A-normaloid if and only if $\omega_A(T) = \|T\|_A$.*

Now, we state the proof of Corollary 2.1.

Proof of Corollary 2.1. (1)⇒(2) Assume that $T \parallel_A S$. Then, by Theorem 2.1 we have $r_A(S^{\sharp_A}T) = \|S^{\sharp_A}T\|_A = \|T^{\sharp_A}S\|_A = \|T\|_A \|S\|_A$. In particular, $S^{\sharp_A}T$ is A-normaloid. So, by Lemma D, $\omega_A(S^{\sharp_A}T) = \|S^{\sharp_A}T\|_A$.

(2)⇒(1) Assume that $\omega_A(S^{\sharp_A}T) = \|S^{\sharp_A}T\|_A = \|T^{\sharp_A}S\|_A = \|T\|_A \|S\|_A$. In particular, by Lemma D, we conclude that $S^{\sharp_A}T$ is A-normaloid. So, by [15, Proposition 4] there exists a sequence of A-unit vectors $\{x_n\}$ such that

$$\lim_{n\to+\infty} \|S^{\sharp_A}Tx_n\|_A = \|S^{\sharp_A}T\|_A \text{ and } \lim_{n\to+\infty} |\langle S^{\sharp_A}Tx_n, x_n\rangle_A| = \omega_A(S^{\sharp_A}T).$$

This implies that

$$\lim_{n\to+\infty} |\langle Tx_n, Sx_n\rangle_A| = \|T\|_A \|S\|_A.$$

Thus, by Theorem C, we conclude that $T \parallel_A S$. □

Next, we investigate the case when an operator $T \in \mathbb{B}_A(\mathcal{H})$ is A-parallel to the identity operator.

Theorem 2.2. *Let $T \in \mathbb{B}_A(\mathcal{H})$. Then the following statements are equivalent:*

(1) $T \parallel_A I$.
(2) $T \parallel_A T^{\sharp_A}$.
(3) $T^{\sharp_A}T \parallel_A T^{\sharp_A}$.

Proof. (1)⇔(2) Assume that $T \parallel_A I$. Then, by Lemma B (v), $\widetilde{T} \parallel I_{\mathbf{R}(A^{1/2})}$. So, $\|\widetilde{T} + \lambda I_{\mathbf{R}(A^{1/2})}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))} = \|\widetilde{T}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))} + 1$, for some $\lambda \in \mathbb{T}$. Then by Lemma 2.3 there exists a state ψ such that such that

$$\begin{aligned}
&\psi\left((\widetilde{T} + \lambda I_{\mathbf{R}(A^{1/2})})^*(\widetilde{T} + \lambda I_{\mathbf{R}(A^{1/2})})\right)\\
&= \|(\widetilde{T} + \lambda I_{\mathbf{R}(A^{1/2})})^*(\widetilde{T} + \lambda I_{\mathbf{R}(A^{1/2})})\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))}\\
&= \|\widetilde{T} + \lambda I_{\mathbf{R}(A^{1/2})}\|^2_{\mathbb{B}(\mathbf{R}(A^{1/2}))}\\
&= \left(\|\widetilde{T}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))} + 1\right)^2.
\end{aligned}$$

So, we see that

$$\begin{aligned}
&\left(\|\widetilde{T}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))} + 1\right)^2\\
&= \psi\left((\widetilde{T} + \lambda I_{\mathbf{R}(A^{1/2})})(\widetilde{T} + \lambda I_{\mathbf{R}(A^{1/2})})^*\right)\\
&= \psi\left(\widetilde{T}(\widetilde{T})^*\right) + \psi(\overline{\lambda}\widetilde{T}) + \psi\left(\lambda(\widetilde{T})^*\right) + 1\\
&\leq \|\widetilde{T}(\widetilde{T})^*\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))} + \|\overline{\lambda}\widetilde{T}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))} + \|\lambda(\widetilde{T})^*\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))} + 1\\
&= \|\widetilde{T}\|^2_{\mathbb{B}(\mathbf{R}(A^{1/2}))} + 2\|\widetilde{T}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))} + 1 = \left(\|\widetilde{T}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))} + 1\right)^2.
\end{aligned}$$

Therefore $\psi(\overline{\lambda}\widetilde{T}) = \psi\left(\lambda(\widetilde{T})^*\right) = \|\widetilde{T}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))}$. This yields that

$$\begin{aligned}\|\widetilde{T}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))} + \|(\widetilde{T})^*\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))} &= \psi\left(\overline{\lambda}\widetilde{T} + \lambda(\widetilde{T})^*\right)\\ &\leq \|\overline{\lambda}\widetilde{T} + \lambda(\widetilde{T})^*\|\\ &= \|\widetilde{T} + \lambda^2(\widetilde{T})^*\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))}\\ &\leq \|\widetilde{T}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))} + \|(\widetilde{T})^*\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))}.\end{aligned}$$

Hence,

$$\|\widetilde{T} + \lambda^2(\widetilde{T})^*\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))} = \|\widetilde{T}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))} + \|(\widetilde{T})^*\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))},$$

in which $\lambda^2 \in \mathbb{T}$. So $\widetilde{T} \parallel (\widetilde{T})^*$. This implies, by Lemma B, that $\widetilde{T} \parallel \widetilde{T^{\sharp_A}}$ which, in turn, yields that $T \parallel_A T^{\sharp_A}$.

Conversely, assume that $T \parallel_A T^{\sharp_A}$ this implies, by Lemma B, that $\widetilde{T} \parallel (\widetilde{T})^*$ which, in turn, yields that

$$\|\widetilde{T} + \lambda(\widetilde{T})^*\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))} = 2\|\widetilde{T}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))},$$

for some $\lambda \in \mathbb{T}$. Since $\widetilde{T} + \lambda(\widetilde{T})^*$ is a normal operator on the Hilbert space $\mathbf{R}(A^{1/2})$, then by Lemma 2.3, there exists a state ψ such that

$$\left|\psi\left(\widetilde{T} + \lambda(\widetilde{T})^*\right)\right| = \|\widetilde{T} + \lambda(\widetilde{T})^*\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))} = 2\|\widetilde{T}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))}.$$

Hence, we obtain

$$2\|\widetilde{T}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))} = \left|\psi\left(\widetilde{T} + \lambda(\widetilde{T})^*\right)\right| \leq 2|\psi(\widetilde{T})| \leq 2\|\widetilde{T}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))}.$$

This implies that $|\psi(\widetilde{T})| = \|\widetilde{T}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))}$. So, there exists a number $\delta \in \mathbb{T}$ such that $\psi(\widetilde{T}) = \delta\|\widetilde{T}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))}$. Thus, we deduce that

$$\begin{aligned}\|\widetilde{T}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))} + 1 &= \psi\big(\overline{\delta}\widetilde{T} + I_{\mathbf{R}(A^{1/2})}\big)\\ &\leq \|\overline{\delta}\widetilde{T} + I_{\mathbf{R}(A^{1/2})}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))}\\ &= \|\widetilde{T} + \delta I_{\mathbf{R}(A^{1/2})}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))} \leq \|\widetilde{T}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))} + 1.\end{aligned}$$

So $\|\widetilde{T} + \delta I_{\mathbf{R}(A^{1/2})}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))} = \|\widetilde{T}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))} + 1$. This immediately implies that $\widetilde{T} \parallel I_{\mathbf{R}(A^{1/2})}$. Hence, $T \parallel_A I$ as required.

(1)⇔(3) Follows from Theorem 2.1.

□

In the next two theorems, we give some characterizations when the A-Davis Wielandt radius of semi-Hilbert space operators attains its upper bound for operators in $\mathbb{B}_{A^{1/2}}(\mathcal{H})$ and $\mathbb{B}_A(\mathcal{H})$, respectively.

Theorem 2.3. *Let $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$. Then, the following assertions are equivalent:*

1. $d\omega_A(T) = \sqrt{\omega_A(T)^2 + \|T\|_A^4}$.
2. $T \parallel_A I$.
3. *T is A-normaloid.*

(4) $\omega_A^2(T)A \geq T^*AT$.

Proof. The equivalences (1) $\Leftrightarrow$ (2) and (2) $\Leftrightarrow$ (3) have been proved in [18]. (3) $\Leftrightarrow$ (4) : By Lemma D, T is A-normaloid if and only if $\omega_A(T) = \|T\|_A$. On the other hand, it be observed that

$$\begin{aligned}\omega_A(T) = \|T\|_A &\Leftrightarrow \|Tx\|_A \leq \omega_A(T)\|x\|_A,\ \forall\, x \in \mathcal{H}\\ &\Leftrightarrow \|Tx\|_A^2 \leq \omega_A(T)^2\|x\|_A^2,\ \forall\, x \in \mathcal{H}\\ &\Leftrightarrow \langle T^*ATx, x\rangle_A \leq \langle \omega_A(T)^2 x, x\rangle_A,\ \forall\, x \in \mathcal{H}\\ &\Leftrightarrow \langle (T^*AT - \omega_A(T)^2A)x, x\rangle \leq 0,\ \forall\, x \in \mathcal{H}\\ &\Leftrightarrow \omega_A^2(T)A \geq T^*AT.\end{aligned}$$

This achieves the proof. □

Theorem 2.4. *Let $T \in \mathbb{B}_A(\mathcal{H})$. The following statements are equivalent:*

(1) $d\omega_A(T) = \sqrt{\omega_A^2(T) + \|T\|_A^4}$.

(2) *There exists a sequence of A-unit vectors $\{x_n\}$ in $\mathcal{H}$ such that*

$$\lim_{n\to+\infty} \left|\langle T^2x_n, x_n\rangle_A\right| = \|T\|_A^2.$$

(3) *There exists a sequence of A-unit vectors $\{x_n\}$ in $\mathcal{H}$ such that*

$$\lim_{n\to+\infty} \left|\langle TT^{\sharp_A}Tx_n, x_n\rangle_A\right| = \|T\|_A^3.$$

(4) $\omega_A(T^2) = \|T\|_A^2$.

Proof. (1) $\Leftrightarrow$ (2) : By Theorem 2.3, we have $d\omega_A(T) = \sqrt{\omega_A^2(T) + \|T\|_A^4}$ if and only if $T \parallel_A I$ which in turn equivalent, by Theorem 2.2, to $T \parallel_A T^{\sharp_A}$. On the other hand, in view of Theorem C, we have $T \parallel_A T^{\sharp_A}$ if and only if there exists a sequence of A-unit vectors $\{x_n\}$ in $\mathcal{H}$ such that

$$\lim_{n\to+\infty} |\langle Tx_n, T^{\sharp_A}x_n\rangle_A| = \|T\|_A\|T^{\sharp_A}\|_A.$$

So, we reach the equivalence (1) $\Leftrightarrow$ (2) since $\|T\|_A = \|T^{\sharp_A}\|_A$.

(1) $\Leftrightarrow$ (3) : By proceeding as above and taking into consideration Theorem 2.2, we deduce that $d\omega_A(T) = \sqrt{\omega_A^2(T) + \|T\|_A^4}$ if and only if $T^{\sharp_A}T \parallel_A T^{\sharp_A}$ which is in turn equivalent, by Theorem 2.2, to the existence of a sequence of A-unit vectors $\{x_n\}$ in $\mathcal{H}$ such that

$$\lim_{n\to+\infty} |\langle T^{\sharp_A}Tx_n, T^{\sharp_A}x_n\rangle_A| = \|T^{\sharp_A}T\|_A\|T^{\sharp_A}\|_A.$$

So, the desired equivalence follows since $\|T^{\sharp_A}\|_A = \|T\|_A = \sqrt{\|T^{\sharp_A}T\|_A}$ and

$$|\langle T^{\sharp_A}Tx_n, T^{\sharp_A}x_n\rangle_A| = |\langle TT^{\sharp_A}Tx_n, x_n\rangle_A|.$$

(1) $\Leftrightarrow$ (4) : If $d\omega_A(T) = \sqrt{\omega_A^2(T) + \|T\|_A^4}$, then by Theorem 2.3 T is A-normaloid. This implies that T is A-spectraloid. So, by [15, Theorem 6] $\omega_A(T^2) = \omega_A^2(T)$. Conversely, assume that $\omega_A(T^2) = \|T\|_A^2$. This implies that the assertion (2) holds and so (1) holds. □

For $x, y \in \mathcal{H}$, we recall from [6] that the A-rank one operators is given by

$$x \otimes_A y \colon \mathcal{H} \to \mathcal{H},\ z \mapsto (x \otimes_A y)(z) := \langle z, y\rangle_A x.$$

A characterization of the A-parallelism of $x \otimes_A y$ and the identity operator is stated as follows.

Theorem 2.5. *Let $x, y \in \mathcal{H}$, the following conditions are equivalent:*

(1) $x \otimes_A y \parallel_A I$.
(2) $d\omega_A(x \otimes_A y) = \sqrt{\omega_A^2(x \otimes_A y) + \|x \otimes_A y\|_A^4}$.
(3) *The vectors $A^{1/2}x$ and $A^{1/2}y$ are linearly dependent.*
(4) *The vectors Ax and Ay are linearly dependent.*

To prove Theorem 2.5 we need the following lemma.

Lemma E. *([6]) Let $x, y \in \mathcal{H}$. Then, the following statement hold:*

(i) $\|x \otimes_A y\|_A = \|x\|_A \|y\|_A$.
(ii) $\omega_A(x \otimes_A y) = \frac{1}{2}\left(|\langle x, y\rangle_A| + \|x\|_A\|y\|_A\right)$.

Now we are ready to prove Theorem 2.5.

Proof of Theorem 2.5. (1) $\Leftrightarrow$ (2) : Follows immediately from Theorem 2.3.

(2) $\Leftrightarrow$ (3) : By the equivalence (2)$\Leftrightarrow$(3) of Theorem 2.3 we infer that

$$d\omega_A(x \otimes_A y) = \sqrt{\omega_A^2(x \otimes_A y) + \|x \otimes_A y\|_A^4} \Leftrightarrow \omega_A(x \otimes_A y) = \|x \otimes_A y\|_A.$$

Moreover, by using Lemma E, we see that

$$\begin{aligned}\omega_A(x \otimes_A y) = \|x \otimes_A y\|_A &\Leftrightarrow \tfrac{1}{2}\left(|\langle x, y\rangle_A| + \|x\|_A\|y\|_A\right) = \|x\|_A\|y\|_A \\ &\Leftrightarrow |\langle x, y\rangle_A| = \|x\|_A\|y\|_A\end{aligned}$$

On the other hand $|\langle x, y\rangle_A| = \|x\|_A\|y\|_A$ if and only if the vectors $A^{1/2}x$ and $A^{1/2}y$ are linearly dependent.

(3) $\Leftrightarrow$ (4) : This equivalence follows immediately since $\mathcal{N}(A) = \mathcal{N}(A^{1/2})$. Hence, the proof is complete. □

3. Further characterizations of A-seminorm-parallelism

Our aim in this section is to give further characterizations of A-seminorm-parallelism via A-Birkhoff-James orthogonality of A-bounded operators. Our first result in this section reads as follows.

Theorem 3.1. *Let $T, S \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$, then the following conditions are equivalent:*

(1) $T \parallel_A S$.
(2) $T \perp_A^{BJ} \|S\|_A T - \lambda\|T\|_A S$, *for some* $\lambda \in \mathbb{T}$.
(3) $S \perp_A^{BJ} \lambda\|T\|_A S - \|S\|_A T$, *for some* $\lambda \in \mathbb{T}$.

In addition, if $\|T\|_A\|S\|_A \neq 0$, then (1) to (3) are also equivalent to

(4) *There exist a sequence of A-unit vectors $\{x_n\}$ in $\mathcal{H}$ and $\lambda \in \mathbb{T}$ such that*

$$\lim_{n\to+\infty} \|Sx_n\|_A = \|S\|_A \quad \textit{and} \quad \lim_{n\to+\infty} \left\| Tx_n - \lambda \frac{\|T\|_A}{\|S\|_A} Sx_n \right\|_A = 0.$$

(5) *There exist a sequence of A-unit vectors $\{x_n\}$ in $\mathcal{H}$ and $\lambda \in \mathbb{T}$ such that*

$$\lim_{n\to+\infty} \|Tx_n\|_A = \|T\|_A \quad \textit{and} \quad \lim_{n\to+\infty} \left\| Sx_n - \lambda \frac{\|S\|_A}{\|T\|_A} Tx_n \right\|_A = 0.$$

In order to prove Theorem 3.1 we need to recall from [27] the following result.

Theorem F. *([27]) Let $T, S \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$. Then, $T \perp_A^{BJ} S$ if and only if there exists a sequence of A-unit vectors $\{x_n\}$ in $\mathcal{H}$ such that*

$$\lim_{n\to+\infty} \|Tx_n\|_A = \|T\|_A \quad \textit{and} \quad \lim_{n\to+\infty} \langle Tx_n, Sx_n\rangle_A = 0.$$

Now, we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. (1) $\Leftrightarrow$ (2) : Assume that $T \parallel_A S$. If $\|S\|_A = 0$, then by using (1.4) it can be seen that the assertion (2) holds. Now, suppose that $\|S\|_A \neq 0$. Since $T \parallel_A S$, then by Lemma 2.1 there exist a sequence of A-unit vectors $\{x_n\}$ in $\mathcal{H}$ and $\lambda \in \mathbb{T}$ such that

$$\lim_{n\to+\infty} \langle Tx_n, Sx_n\rangle_A = \lambda \|T\|_A \|S\|_A.$$

So, by Remark 2.1 $\lim\limits_{n\to+\infty} \|Tx_n\|_A = \|T\|_A$. Furthermore, we see that

$$\begin{aligned} &\lim_{n\to+\infty} \langle Tx_n, (\|S\|_A T - \lambda \|T\|_A S)x_n\rangle_A \\ &= \lim_{n\to+\infty} \|S\|_A \|Tx_n\|_A^2 - \overline{\lambda} \|T\|_A \langle Tx_n, Sx_n\rangle_A \\ &= \|S\|_A \|T\|_A^2 - \|T\|_A^2 \|S\|_A = 0. \end{aligned}$$

Thus, in view of Theorem F, the second assertion holds. Conversely, assume $T \perp_A^{BJ} \|S\|_A T - \lambda \|T\|_A S$, for some $\lambda \in \mathbb{T}$. If $\|T\|_A = 0$, then obviously $T \parallel_A S$. Suppose that $\|T\|_A \neq 0$. By Theorem F, there exists a sequence of A-unit vectors $\{y_n\}$ in $\mathcal{H}$ such that

$$\lim_{n\to+\infty} \|Ty_n\|_A = \|T\|_A \quad \text{and} \quad \lim_{n\to+\infty} \langle Ty_n, (\|S\|_A T - \lambda \|T\|_A S)y_n\rangle_A = 0.$$

Then, we deduce that

$$\lim_{n\to+\infty} \langle Ty_n, Sy_n\rangle_A = \frac{\lambda}{\|T\|_A} \lim_{n\to+\infty} \|S\|_A \|Ty_n\|_A^2 = \lambda \|T\|_A \|S\|_A.$$

(1) $\Leftrightarrow$ (3) : The proof is analogous to the previous equivalence by changing the roles between T and S.
(1) $\Leftrightarrow$ (4) : By Lemma 2.1 and Remark 2.1, there exist a sequence of A-unit

vectors $\{x_n\}$ in $\mathcal{H}$ and $\lambda \in \mathbb{T}$ such that $\lim\limits_{n\to+\infty} \langle Tx_n, Sx_n\rangle = \lambda\|T\|_A\|S\|_A$, $\lim\limits_{n\to+\infty} \|Tx_n\|_A = \|T\|_A$ and $\lim\limits_{n\to+\infty} \|Sx_n\|_A = \|S\|_A$. So, since

$$\begin{aligned}&\left\|Tx_n - \lambda\frac{\|T\|_A}{\|S\|_A}Sx_n\right\|_A^2 \\ &= \|Tx_n\|_A^2 - \overline{\lambda}\frac{\|T\|_A}{\|S\|_A}\langle Tx_n, Sx_n\rangle_A - \lambda\frac{\|T\|_A}{\|S\|_A}\langle Sx_n, Tx_n\rangle_A + \frac{\|T\|_A^2}{\|S\|_A^2}\|Sx_n\|_A^2,\end{aligned}$$

then we deduce that $\lim\limits_{n\to+\infty} \left\|Tx_n - \lambda\frac{\|T\|_A}{\|S\|_A}Sx_n\right\|_A^2 = 0$. Conversely, suppose that (4) holds. Then, we see that

$$\begin{aligned}\|S\|_A + \|T\|_A &\geq \|T + \lambda S\|_A \\ &\geq \|Tx_n + \lambda Sx_n\|_A \\ &= \left\|(Tx_n - \lambda\frac{\|T\|_A}{\|S\|_A}Sx_n) - (-\lambda Sx_n - \lambda\frac{\|T\|_A}{\|S\|_A}Sx_n)\right\|_A \\ &\geq \left\|\lambda Sx_n + \lambda\frac{\|T\|_A}{\|S\|_A}Sx_n\right\|_A - \left\|Tx_n - \lambda\frac{\|T\|_A}{\|S\|_A}Sx_n\right\|_A \\ &= (\|S\|_A + \|T\|_A)\frac{\|Sx_n\|_A}{\|S\|_A} - \left\|Tx_n - \lambda\frac{\|T\|_A}{\|S\|_A}Sx_n\right\|_A.\end{aligned}$$

By taking limits, we get $\|S\|_A + \|T\|_A = \|T + \lambda S\|_A$. Then $T \parallel_A S$.
(1) $\Leftrightarrow$ (5) : The proof is analogous to the previous equivalence by changing the roles between T and S. □

Corollary 3.1. *Let $T \in \mathbb{B}_A(\mathcal{H})$. Then the following statements are equivalent:*

1. $T \parallel_A I$.
2. $T^p \parallel_A I$ *for every* $p \in \mathbb{N}$.
3. $T^p \parallel_A (T^{\sharp_A})^p$ *for every* $p \in \mathbb{N}$.

Proof. (1)⇒(2) Assume that $T \parallel_A I$. Then, by Theorem 3.1, there exists a sequence of A-unit vectors $\{x_n\}$ in $\mathcal{H}$ and $\lambda \in \mathbb{T}$ such that

$$\lim_{n\to+\infty} \left\|Tx_n - \lambda\|T\|_A x_n\right\|_A = 0.$$

For every $i \in \mathbb{N}$ we have

$$\begin{aligned}&\left\|\left(T^{i+1} - \lambda^{i+1}\|T\|_A^{i+1}I\right)x_n\right\|_A \\ &= \left\|T\left(T^i - \lambda^i\|T\|_A^iI\right)x_n + \lambda^i\|T\|_A^i\left(T - \lambda\|T\|_AI\right)x_n\right\|_A \\ &\leq \|T\|_A\left\|(T^i - \lambda^i\|T\|_A^iI)x_n\right\|_A + \|T\|_A^i\left\|(T - \lambda\|T\|_AI)x_n\right\|_A.\end{aligned}$$

So, by induction, it can be shown that for every $p \in \mathbb{N}$ we have

$$\lim_{n\to+\infty} \left\|(T^p - \lambda^p\|T\|_A^pI)x_n\right\|_A = 0. \tag{3.1}$$

This implies, by Lemma B, that

$$\lim_{n\to+\infty}\left\| \left((\widetilde{T})^p - \lambda^p\|T\|_A^p I_{\mathbf{R}(A^{1/2})}\right) Ax_n\right\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))} = 0,$$

for every $p\in\mathbb{N}$. Hence, $\lambda^p\|T\|_A^p \in \sigma_a\left((\widetilde{T})^p\right)$. So, we obtain

$$\|\widetilde{T}\|^p_{\mathbb{B}(\mathbf{R}(A^{1/2}))} \leq r\left((\widetilde{T})^p\right) \leq \left\|(\widetilde{T})^p\right\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))} \leq \left\|\widetilde{T}\right\|^p_{\mathbb{B}(\mathbf{R}(A^{1/2}))}.$$

Thus, an application of Lemma B(i) gives $\|T\|_A^p = \|T^p\|_A$. So, by taking into consideration (3.1), we get

$$\lim_{n\to+\infty}\left\|(T^p - \lambda^p\|T^p\|_A I)x_n\right\|_A = 0,$$

for every $p\in\mathbb{N}$. Therefore, by Theorem 3.1, we get $T^p \parallel_A I$.

Now, the implications (2)⇒(3) and (3)⇒(1) follow immediately by using the equivalences of Theorem 2.2. □

Remark 3.1. *Notice that the equivalence* (1) ⇔ (2) *in Corollary 3.1 holds also for A-bounded operators.*

A special case of A-seminorm-parallelism between an A-bounded operator $T\in\mathbb{B}_{A^{1/2}}(\mathcal{H})$ and the identity operator, is the following equation:

$$\|T+I\|_A = \|T\|_A + 1. \tag{3.2}$$

If $T\in\mathbb{B}_{A^{1/2}}(\mathcal{H})$ and satisfies (3.2), we shall say that T satisfies the A-Daugavet equation. We remind here that the first person who study (3.2) for $A=I$ was I. K. Daugavet [11]. The equation is one useful property in solving a variety of problems in approximation theory. Abramovich et al. [1] proved that $T\in\mathbb{B}(\mathcal{H})$ satisfies the I-Daugavet equation (respect to the uniform norm) if and only if $\|T\|$ lies in the approximate point spectrum of T.

In the following theorem we shall characterize A-bounded operators which satisfy the A-Daugavet equation.

Theorem 3.2. *Let $T\in\mathbb{B}_{A^{1/2}}(\mathcal{H})$. Then the following conditions are equivalent:*

(1) *T satisfies the A-Daugavet equation, i.e.* $\|T+I\|_A = \|T\|_A + 1$.
(2) $\|T\|_A \in \overline{W_A(T)}$.
(3) $I \perp_A^{BJ} \|T\|_A I - T$.
(4) $T \perp_A^{BJ} T - \|T\|_A I$.

Proof. (2) ⇒ (1) Assume that $\|T\|_A \in \overline{W_A(T)}$. Then, there exits a sequence of A-unit vectors $\{x_n\}$ in $\mathcal{H}$ such that $\lim\limits_{n\to+\infty}\langle Tx_n, x_n\rangle_A = \|T\|_A$. Thus

$$\lim_{n\to+\infty}\Re(\langle Tx_n, x_n\rangle)_A = \|T\|_A. \tag{3.3}$$

Further, since

$$\begin{aligned}\|T\|_A^2 + 2|\langle Tx_n, x_n\rangle_A| + 1 &\leq \|T\|_A^2 + 2\|Tx_n\|_A + 1\\ &\leq \|T\|_A^2 + 2\|T\|_A + 1 = (\|T\|_A + 1)^2,\end{aligned}$$

for all $n \in \mathbb{N}$, then we get

$$\lim_{n\to+\infty} \|Tx_n\|_A = \|T\|_A. \tag{3.4}$$

Hence, by using (3.3) together with (3.4) we see that

$$\begin{aligned}(\|T\|_A + 1)^2 &= \lim_{n\to+\infty} \|Tx_n\|_A^2 + 2\lim_{n\to+\infty} \Re(\langle Tx_n, x_n\rangle_A) + 1\\ &= \lim_{n\to+\infty} \|(T+I)x_n\|_A^2 \leq \|T+I\|_A^2 \leq (\|T\|_A + 1)^2.\end{aligned}$$

So $\|T+I\|_A = \|T\|_A + 1$.

(1) $\Rightarrow$ (2) Suppose that $\|T+I\|_A = \|T\|_A + 1$. Then, by (1.3) there exists a sequence of A-unit vectors $\{x_n\}$ in $\mathcal{H}$ such that

$$\lim_{n\to+\infty} \|Tx_n + x_n\|_A = \|T\|_A + 1. \tag{3.5}$$

Since

$$\|Tx_n + x_n\|_A \leq \|Tx_n\|_A + 1 \leq \|T\|_A + 1,$$

then, by using (3.5), we conclude that

$$\lim_{n\to+\infty} \|Tx_n\|_A = \|T\|_A. \tag{3.6}$$

On the other hand, since

$$\|Tx_n + x_n\|_A^2 = \|Tx_n\|_A^2 + 1 + 2\Re(\langle Tx_n, x_n\rangle_A),$$

for all $n \in \mathbb{N}$, then it follows from (3.5) together with (3.6) that

$$\lim_{n\to+\infty} \Re(\langle Tx_n, x_n\rangle_A) = \|T\|_A, \tag{3.7}$$

for all $n \in \mathbb{N}$. Further, if $\Im(z)$ denotes the imaginary part of $z \in \mathbb{C}$, then for every $n \in \mathbb{N}$, we see that

$$\Re^2(\langle Tx_n, x_n\rangle_A) \leq \Re^2(\langle Tx_n, x_n\rangle_A) + \Im^2(\langle Tx_n, x_n\rangle_A) = |\langle Tx_n, x_n\rangle_A|^2 \leq \|T\|_A^2.$$

So, by (3.7), we infer that $\lim_{n\to+\infty} \Im(\langle Tx_n, Sx_n\rangle_A) = 0$. This yields, through (3.7), that

$$\lim_{n\to+\infty} \langle Tx_n, x_n\rangle_A = \|T\|_A.$$

Thus, we conclude that $\|T\|_A \in \overline{W_A(T)}$.

(1) $\Leftrightarrow$ (3) Assume that T satifies the A-Daugavet equation. Then, by the equivalence between (1) and (2), we have $\|T\|_A \in \overline{W_A(T)}$. So, there exists a sequence of A-unit vectors $\{x_n\}$ in $\mathcal{H}$ satisfying

$$\lim_{n\to+\infty} \langle Tx_n, x_n\rangle_A = \|T\|_A. \tag{3.8}$$

This implies that

$$\lim_{n\to+\infty} \|Ix_n\|_A = \|I\|_A = 1 \quad \text{and} \quad \lim_{n\to+\infty} \langle (T - \|T\|_A I)x_n, x_n\rangle_A = 0,$$

then by Theorem F, we have $I \perp_A^{BJ} \|T\|_A I - T$. The converse is analogous.
(1) $\Leftrightarrow$ (4) Assume that T satifies the A-Daugavet equation. Let $\{x_n\}$ a sequence of A-unit vectors in $\mathcal{H}$ satisfying (3.8). Then

$$\|T\|_A \geq \|Tx_n\|_A \geq |\langle Tx_n, x_n\rangle_A| \geq \|T\|_A - \varepsilon,$$

for any $\varepsilon > 0$ and n large enough. Hence, $\lim\limits_{n\to+\infty} \|Tx_n\|_A = \|T\|_A$. Futhermore,

$$\lim_{n\to+\infty} \langle Tx_n, (T - \|T\|_A I)x_n\rangle_A = \lim_{n\to+\infty} \|Tx_n\|_A^2 - \|T\|_A \langle Tx_n, x_n\rangle_A = 0.$$

So, by Theorem F, we deduce that $T \perp_A^{BJ} T - \|T\|_A I$. Conversely, assume that $T \perp_A^{BJ} T - \|T\|_A I$. If $\|T\|_A = 0$, then by using (1.4) we see that the assertion (1) holds trivially. Now, suppose that $\|T\|_A \neq 0$. By Theorem F, there exists a sequence of A-unit vectors $\{y_n\}$ in $\mathcal{H}$ such that

$$\lim_{n\to+\infty} \|Ty_n\|_A = \|T\|_A \quad \text{and} \quad \lim_{n\to+\infty} \langle Ty_n, (T - \|T\|_A I)y_n\rangle_A = 0.$$

So, it follows that

$$\lim_{n\to+\infty} \langle Ty_n, y_n\rangle_A = \frac{1}{\|T\|_A} \lim_{n\to+\infty} \|Ty_n\|_A^2 = \|T\|_A,$$

i.e. $\|T\|_A \in \overline{W_A(T)}$. Hence, by the equivalence (1)$\Leftrightarrow$(2), the assertion (1) holds. Therefore, the proof is complete. ☐

4. A-Bikhorff-James orthogonality and distance formulas

We begin this section by recalling from [27] the following definition.

Definition 4.1. *Let $T, S \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$. The A-distance between T and S, denoted by $d_A(T, \mathbb{C}S)$, is defined as*

$$d_A(T, \mathbb{C}S) := \inf_{\gamma\in\mathbb{C}} \|T + \gamma S\|_A.$$

Our first result in this section provides an upper bound for the nonnegative quantity $\|T\|_A^2 - \omega_A^2(T)$, with $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$ related to $d_A(T, \mathbb{C}I)$.

Theorem 4.1. *Let $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$. Then,*

$$\|T\|_A^2 - \omega_A^2(T) \leq d_A^2(T, \mathbb{C}I). \tag{4.1}$$

Proof. Notice first that for any $a, b \in \mathcal{H}$ with $b \neq 0$, we have

$$\inf_{\lambda\in\mathbb{C}} \|a - \lambda b\|^2 = \frac{\|a\|^2\|b\|^2 - |\langle a, b\rangle|^2}{\|b\|^2}.$$

This implies that

$$\|a\|^2\|b\|^2 - |\langle a, b\rangle|^2 \leq \|b\|^2\|a - \lambda b\|^2, \tag{4.2}$$

for any $a, b \in \mathcal{H}$ and $\lambda \in \mathbb{C}$. Let $x, y \in \mathcal{H}$ and $\lambda \in \mathbb{C}$. By choosing $a = A^{1/2}x$ and $b = A^{1/2}y$ in (4.2), we obtain

$$\|x\|_A^2\|y\|_A^2 - |\langle x, y\rangle_A|^2 \leq \|y\|_A^2\|x - \lambda y\|_A^2, \tag{4.3}$$

Now, by choosing in (4.3) $x = Tz$ and $y = z$ with $z \in \mathcal{H}$, $\|z\|_A = 1$, we get

$$\|Tz\|_A^2 - |\langle Tz, z\rangle_A|^2 \leq \|Tz - \lambda z\|_A^2,$$

By taking the supremum over all $z \in \mathcal{H}$ with $\|z\|_A = 1$, we obtain

$$\|T\|_A^2 - \omega_A^2(T) \leq \inf_{\lambda\in\mathbb{C}} \|T - \lambda I\|_A^2.$$

This finishes the proof of the theorem. □

Remark 4.1. *Notice that the third author proved in* [17, Theorem 2.22.] *that for every $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$ we have*

$$\omega_A^2(T) \leq \frac{1}{2}(\omega_A(T^2) + \|T\|_A^2). \tag{4.4}$$

So, by combining (4.4) *together with* (4.1)*, we obtain*

$$\omega_A^2(T) - \omega_A(T^2) \leq \frac{1}{2}\left(\|T\|_A^2 - \omega_A(T^2)\right) \leq \|T\|_A^2 - \omega_A(T^2) \leq d_A^2(T, \mathbb{C}I),$$

for any $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$.

Next, we recall from [27] that the A-minimum modulus of an operator $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$ is given by

$$m_A(T) = \inf\Big\{\|Tx\|_A\,;\ x \in \mathcal{H},\ \ \|x\|_A = 1\Big\}.$$

This concept is useful in characterizing the A-Bikhorff-James orthogonality in $\mathbb{B}_{A^{1/2}}(\mathcal{H})$. More precisely, we have the following result.

Theorem G. *(*[27, Theorem 3.2]*) Let $T, S \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$ with $m_A(S) > 0$. Then there exists a unique $t_0 \in \mathbb{C}$ such that*

$$\|(T - t_0S) + \gamma S\|_A^2 \geq \|(T - t_0S)\|_A^2 + |\gamma|^2 m_A^2(S), \tag{4.5}$$

for every $\gamma \in \mathbb{C}$. Futhermore, such t_0 satisfies the following property

$$\|T - t_0S\|_A = d_A(T, \mathbb{C}S).$$

Inspiring from the definition of center of mass in the case of Hilbert space operators due to Barra and Bouzmagour (see [5]), we define the following new concept.

Definition 4.2. *Given $T, S \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$ with $m_A(S) > 0$. The A-center of mass of T relatively to S, denoted by $c_A(T, S)$, is defined to be the unique $t_0 \in \mathbb{C}$ specified in Theorem G. That is*

$$\|T - c_A(T, S)S\|_A = d_A(T, \mathbb{C}S).$$

For a given $T, S \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$ with $m_A(S) > 0$, Zamani proved in [27, Theorem 3.4] that

$$d_A^2(T, \mathbb{C}S) = \sup_{\|x\|_A=1}\left(\|Tx\|_A^2 - \frac{|\langle Tx, Sx\rangle_A|^2}{\|Sx\|_A^2}\right). \tag{4.6}$$

One of the methods to compute the center of mass of an operator is Williams's theorem [25]. However, it is not usually easy to determine the exact value of it even in the finite dimensional case. In what follows, we investigate how to determine explicitly the number $c_A(T,S)$.

Theorem 4.2. *Let $T,S\in\mathbb{B}_{A^{1/2}}(\mathcal{H})$ with $m_A(S)>0$. Then*

$$c_A(T,S)=\lim_{n\to+\infty}\frac{\langle Tx_n,Sx_n\rangle_A}{\|Sx_n\|_A^2},$$

where $\{x_n\}$ be a sequence of A-unit vectors, approximating the supremum in (4.6).

Proof. By the hypothesis, $m_A(S)>0$, we can conclude that $\|Sx\|_A\geq m_A(S)>0$ for all $x\in\mathcal{H}$ with $\|x\|_A=1$. For sake of simplicity we denote $c_A=c_A(T,S)$. Let $\{x_n\}$ be a sequence of A-unit vectors, approximating the supremum in (4.6). Then

$$\begin{aligned}
&\left|\frac{\langle Tx_n,Sx_n\rangle_A}{\|Sx_n\|_A}-c_A\|Sx_n\|_A\right|^2\\
&=\frac{|\langle Tx_n,Sx_n\rangle_A|^2}{\|Sx_n\|_A^2}-2\Re(\langle Tx_n,c_ASx_n\rangle_A)+|c_A|^2\|Sx_n\|_A^2\\
&=\|(T-c_AS)x_n\|_A^2-\|Tx_n\|_A^2+\frac{|\langle Tx_n,Sx_n\rangle_A|^2}{\|Sx_n\|_A^2}\\
&\leq\|(T-c_AS)\|_A^2-\|Tx_n\|_A^2+\frac{|\langle Tx_n,Sx_n\rangle_A|^2}{\|Sx_n\|_A^2}.
\end{aligned}$$

As $\|Sx\|_A\geq m_A(S)$ for any $\|x\|_A=1$, we obtain the following inequality

$$\left|\frac{\langle Tx_n,Sx_n\rangle_A}{\|Sx_n\|_A^2}-c_A\right|\leq\frac{1}{m_A(S)}\left|\frac{\langle Tx_n,Sx_n\rangle_A}{\|Sx_n\|_A}-c_A\|Sx_n\|_A\right|\xrightarrow{n\to+\infty}0.$$

□

Two particular cases of the special interest are considered in the next statement, first one when $S=T^{\sharp_A}$ and later when in addition T is A-normal.

Corollary 4.1. *Let $T\in\mathbb{B}_A(\mathcal{H})$ with $m_A(T^{\sharp_A})>0$. Then*

$$c_A(T,T^{\sharp_A})=\lim_{n\to+\infty}\frac{\langle T^2x_n,x_n\rangle_A}{\|T^{\sharp_A}x_n\|_A^2},$$

where $\{x_n\}$ be a sequence of A-unit vectors, approximating the supremum in (4.6). In addition, if T is A-normal, then $|c_A(T,T^{\sharp_A})|\leq 1$.

The following theorem is a natural generalization of a result due to Fujii and Prasanna in [19].

Theorem 4.3. *Let $T\in\mathbb{B}_{A^{1/2}}(\mathcal{H})$. Then*

$$W_A(T)\subseteq D\Big(c_A(T,I),d_A(T,\mathbb{C}I)\Big),$$

where $D(\lambda_0,r_0)$ denotes the closed disc centered at λ_0 and with radius r_0.

Proof. We split the proof in two cases.

Case 1: $c_A(T,I)=0$ i.e. $d_A(T,\mathbb{C}I)=\|T\|_A$. Then for any $x\in\mathcal{H}$ with $\|x\|_A=1$, we have

$$|\langle Tx,x\rangle_A|\leq \omega_A(T)\leq \|T\|_A=d_A(T,\mathbb{C}I). \tag{4.7}$$

Case 2: $c_A(T,I)\neq 0$ i.e. $d_A(T,\mathbb{C}I)=\|T-c_A(T,I)I\|_A$. Let us consider $T_0:=T-c_A(T,I)I$. Then $T_0\in\mathbb{B}_{A^{1/2}}(\mathcal{H})$ and $c_A(T_0,I)=0$. Applying (4.7), we obtain for any $x\in\mathcal{H}$, $\|x\|_A=1$

$$|\langle Tx,x\rangle_A-c_A(T,I)|=|\langle T_0x,x\rangle_A|\leq \|T_0\|_A=d_A(T,\mathbb{C}I).$$

This completes the proof. □

Proposition 4.1. *Let $T\in\mathbb{B}_{A^{1/2}}(\mathcal{H})$. Then*

$$d_A(T,\mathbb{C}I)\leq \|T\|_A d_A(I,\mathbb{C}T). \tag{4.8}$$

Proof. Let $x\in\mathcal{H}$ with $\|x\|_A=1$. One observes that

$$\alpha_A(T)\|Tx\|_A\leq |\langle Tx,x\rangle_A|,$$

where $\alpha_A(T)=\inf\left\{\frac{|\langle Ty,y\rangle_A|}{\|Ty\|_A}:\|Ty\|_A\neq 0,\|y\|_A=1\right\}$ if $\|T\|_A\neq 0$ or $\alpha_A(T)=0$ if $\|T\|_A=0$. Thus, we see that

$$\|Tx\|_A^2-|\langle Tx,x\rangle_A|^2\leq \left(1-\alpha_A^2(T)\right)\|Tx\|_A^2\leq d_A^2(I,\mathbb{C}T)\|Tx\|_A^2.$$

Now, calculating the supremum of the both sides, over all $x\in\mathcal{H}$ with $\|x\|_A=1$, we complete the proof. □

Remark 4.2. *By combining (4.1) together with (4.8), we obtain*

$$\|T\|_A^2-\omega_A^2(T)\leq d_A^2(T,\mathbb{C}I)\leq \|T\|_A^2 d_A^2(I,\mathbb{C}T).$$

Corollary 4.2. *Let $T\in\mathbb{B}_{A^{1/2}}(\mathcal{H})$. If $T\perp_A^{BJ} I$, then $I\perp_A^{BJ} T$.*

Proof. By (4.8), we have

$$\|T\|_A=d_A(T,\mathbb{C}I)\leq \|T\|_A d_A(I,\mathbb{C}T).$$

So, if $\|T\|_A\neq 0$, then $1\leq d_A(I,\mathbb{C}T)\leq \|I\|_A=1$, i.e. $d_A(I,\mathbb{C}T)=\|I\|_A=1$.

On the other hand, if $\|T\|_A=0$ then $\|Tx\|_A=0$ for all $x\in\mathcal{H}$, $\|x\|_A=1$. From [27, Theorem 3.4], we have that

$$d_A^2(I,\mathbb{C}T)=\sup\{\|Ix\|_A^2\,;\;\|x\|_A=1\}=1=\|I\|_A.$$

In conclusion, in both cases, we obtain that $I\perp_A^{BJ} T$. □

The converse of the previous result is false in general, as we see in the next example

Example 4.1. *Consider in $\mathcal{H}=\mathbb{C}^3$ with the usual uniform norm and let $\{e_1,e_2,e_3\}$ be the canonical basis for $\mathcal{H}$. Let $A=\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$. Then $A=P_{\mathcal{M}}$*

the orthogonal projection on $\mathcal{M} = gen\{e_1, e_2\}$ *and* $A^2 = A^* = A$. *Now, consider* $T = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$. *Let* $x = \alpha e_1 + \beta e_2 + \gamma e_3 \in \mathcal{H}$ *then*

$$\|x\|_A^2 = \|(\alpha, \beta, \gamma)\|_A^2 = \langle x, x\rangle_A = \langle Ax, Ax\rangle = \|Ax\|^2 = |\alpha|^2 + |\beta|^2 = \|(\alpha, \beta)\|^2.$$

Observe that $\|(\alpha, \beta, \gamma)\|_A^2 = 1$ *if and only if* $\|(\alpha, \beta)\|^2 = 1$. *Further, we have*

$$\begin{aligned}\|T\|_A^2 &= \sup\{\|Tx\|_A^2 : x \in \mathbb{C}^3, \|x\|_A = 1\} = \sup\{\|ATx\|^2 : x \in \mathbb{C}^3, \|x\|_A = 1\} \\ &= \sup\{\|\overline{T}x\|^2 : \overline{x} \in \mathbb{C}^2, \|\overline{x}\| = 1\} = \|\overline{T}\|^2 = 4,\end{aligned}$$

where $\overline{T} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \in \mathbb{B}(\mathbb{C}^2)$. *If* I_n *denotes the identity operator in* $\mathbb{B}(\mathbb{C}^n)$, *then*

$$\inf_{\lambda \in \mathbb{C}} \|T - \lambda I_3\|_A = \inf_{\lambda \in \mathbb{C}} \|\overline{T} - \lambda I_2\| = \frac{3}{2} < \|T\|_A = 2,$$

i.e. T *is not* A*-Birkhoff-James to* I_3. *On the other hand,*

$$\inf_{\lambda \in \mathbb{C}} \|I_3 - \lambda T\|_A = \inf_{\lambda \in \mathbb{C}} \|I_2 - \lambda \overline{T}\| = 1 = \|I_3\|_A = 1,$$

that is $I_3 \perp_A^{BJ} T$.

The following result relates A-Birkhoff-James orthogonality with the attainment of the lower bound of the A-Davis-Wielandt radius.

Theorem 4.4. *Let* $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$ *such that* $d\omega_A(T) = \max\{\omega_A(T), \|T\|_A^2\}$. *Then* $T \perp_A^{BJ} I$.

Proof. We separate in two different cases.

Case 1: Suppose $d\omega_A(T) = \|T\|_A^2$ and take a sequence of A-unit vectors $\{y_n\}_{n\in\mathbb{N}}$ such that $\lim\limits_{n\to+\infty} \|Ty_n\|_A^2 = \|T\|_A^2$. Then

$$\|Ty_n\|_A^2 \leq \sqrt{\left|\langle Ty_n, y_n\rangle_A\right|^2 + \|Ty_n\|_A^4} \leq d\omega_A(T) = \|T\|_A^2.$$

Therefore, we infer that $\lim\limits_{n\to+\infty} \left|\langle Ty_n, y_n\rangle_A\right|^2 = 0$. This is equivalent, by Theorem F, to $T \perp_{BJ}^A I$.

Case 2: Suppose $d\omega_A(T) = \omega_A(T)$ and take a sequence of A-unit vectors $\{z_n\}_{n\in\mathbb{N}}$ such that $\lim\limits_{n\to+\infty} |\langle Tz_n, z_n\rangle_A| = \omega_A(T)$. Then

$$|\langle Tz_n, z_n\rangle_A| \leq \sqrt{\left|\langle Tz_n, z_n\rangle_A\right|^2 + \|Tz_n\|_A^4} \leq d\omega_A(T) = \omega_A(T),$$

therefore, $\lim\limits_{n\to+\infty} \|Tz_n\|_A^4 = 0$. But

$$|\langle Tz_n, z_n\rangle_A| \leq \|Tz_n\|_A \to 0,$$

thus $\omega_A(T) = 0$ and $\|T\|_A = 0 \leq \|T + \lambda I\|_A$ for every $\lambda \in \mathbb{C}$. □

We arrive to the next conclusion as a combination of Corollary 4.2 and Theorem 4.4.

Corollary 4.3. *Let $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$ such that $d\omega_A(T) = \max\{\omega_A(T), \|T\|_A^2\}$. Then $T \perp_A^{BJ} I$ and $I \perp_A^{BJ} T$.*

Remark 4.3. *If $T = x \otimes_A y$ with $\|x\|_A, \|y\|_A \neq 0$, the attainment of the lower bound of $d\omega_A(T)$ implies that $x \perp_A y$.*

Indeed, first of all, if $u, v \in \mathcal{H}$, then one may observe that

$$\frac{1}{2}\left(|\langle u, v\rangle_A| + \|u\|_A\|v\|_A\right) \leq \|u\|_A\|v\|_A.$$

So, if $d\omega_A(T)$ attains its lower bound we may assume that $dw_A(T) = \|T\|_A^2 = \|x\|_A^2\|y\|_A^2$. Then, we see that

$$\begin{aligned}\left|\langle T\frac{y}{\|y\|_A}, \frac{y}{\|y\|_A}\rangle_A\right| &= \left|\langle x, \frac{y}{\|y\|_A}\rangle_A\langle \frac{y}{\|y\|_A}, y\rangle_A\right| = \left|\frac{1}{\|y\|_A^2}\langle x, y\rangle_A\|y\|_A^2\right| \\ &= |\langle x, y\rangle_A|,\end{aligned}$$

and

$$\left\|T\frac{y}{\|y\|_A}\right\|_A^4 = \frac{1}{\|y\|_A^4}\left\|\langle y, y\rangle_A x\right\|_A^4 = \|y\|_A^4\|x\|_A^4.$$

Therefore, we have

$$\sqrt{\left|\langle T\frac{y}{\|y\|_A}, \frac{y}{\|y\|_A}\rangle_A\right|^2 + \left\|T\frac{y}{\|y\|_A}\right\|_A^4} = \sqrt{|\langle x, y\rangle_A|^2 + \|y\|_A^4\|x\|_A^4}.$$

In particular, we obtain that

$$d\omega_A^2(T) \geq |\langle x, y\rangle_A|^2 + \|y\|_A^4\|x\|_A^4.$$

Since by hypothesis, $dw_A(T) = \|x\|_A^2\|y\|_A^2$, then it follows that $\|x\|_A^4\|y\|_A^4 = d\omega_A^2(T) \geq |\langle x, y\rangle_A|^2 + \|x\|_A^4\|y\|_A^4$. This clearly forces $\langle x, y\rangle_A = 0$. Hence, $x \perp_A y$.

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